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ME451 Laboratory
Experiment #5

Sinusoidal Response of a 2nd Order Torsional Mass-Spring Damper System

Important: Bring your short form from lab #2 to the lab. You will need them for comparison.

ME451 Laboratory Manual Pages, Last Revised: October 31, 2007
Send comments to: Dr. Clark Radcliffe, Professor

Reference:

C.L. Phillips and R.D. Harbor, *Feedback Control Systems*, Prentice Hall, 4th Ed.
Section 4.2, pp. 121-124: Time Response of Second-Order Systems
Section 4.4, pp. 129-132: Frequency Response of Systems
Appendix B, pp. 635-650: Laplace Transform

1. Objective

The response of a linear system to a sinusoidal input is useful for predicting its behavior for arbitrary periodic inputs, but more importantly, for compensator design. For second-order systems, the sinusoidal response depends primarily on the natural frequency, ω_n, and the damping ratio, ζ. Both ω_n and ζ are functions of system parameters, both physical and control parameters. In this experiment we will alter the ω_n and ζ values by changing the feedback control gains. The objective of this experiment is to investigate the relationship between ω_n and ζ and the frequency response of the system, as well as the relationship between the feedback gains and ω_n and ζ. The second-order system we choose for this experiment is a torsional mass-spring-damper system, with torque as input and angular displacement as output. We obtain the transfer function of the system and identify specific parameters of the system that affect sinusoidal response. Specifically, we identify parameters that affect the natural frequency and the damping ratio. We vary these parameters to experimentally verify the change in sinusoidal response.

2. Background

2.1. Second-order systems

The standard form of transfer function of a second-order system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{1}$$

where Y (s) and U(s) are the Laplace transforms of the output and input variables, respectively, ω_n is the natural frequency, and ζ is the damping ratio. For a sinusoidal input

$$u(t) = A\sin(\omega * t), \qquad U(s) = \frac{A\omega}{s^2 + \omega^2}$$

the response of the system, in Laplace domain, can be written as

$$Y(s) = \frac{KA\omega\omega_n^2}{(s^2 + \omega^2)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Assuming poles of G(s) are in the left-half plane, the steady state response of the system (after transients have decayed) can be written as

$$y(t) = A\left|G(j\omega)\right|\sin(\omega * t + \varphi), \qquad \varphi \equiv \angle G(j\omega) \tag{2}$$

It is clear from Eq.(2) that a sinusoidal input produces a sinusoidal output. The amplitude of the output is scaled by a factor of $\left|G(j\omega)\right|$ and the phase lags behind the input by $\angle G(j\omega)$.

For the standard second-order system in Eq.(1), given the values of ω_n and ζ, the "gain" $|G(j\omega)|$ and the "phase" $\angle G(j\omega)$ can be expressed as a function of ω, as follows

$$|G(j\omega)| = \frac{K}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}} \; ; \qquad r \equiv (\frac{\omega}{\omega_n}) \qquad (3)$$

$$\varphi = \angle G(j\omega) = -\arctan\left[\frac{2\zeta * r}{1-r^2}\right] \text{ (gives units of radians)}$$

On a logarithmic scale, they can be plotted to generate what are known as gain and phase plots, or Frequency Response diagrams. The Frequency Response diagrams for a standard second-order system are plotted as a function of the frequency ratio $r \equiv (\frac{\omega}{\omega_n})$, for different values of ζ in Fig.1.

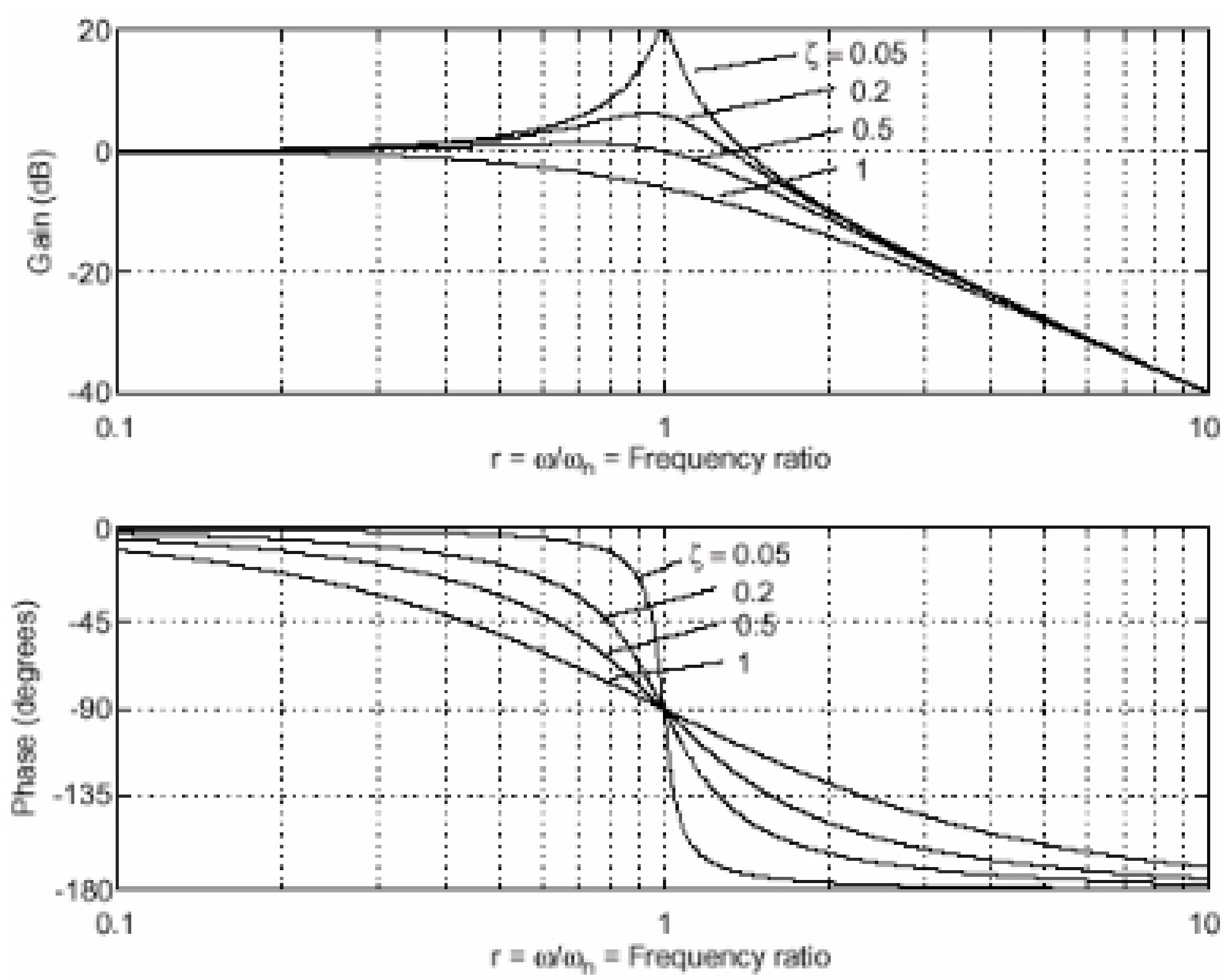

Figure 1. Frequency Response diagram for standard second-order system

2.2. Torsional mass-spring-damper system

Recall the torsional mass-spring-damper system in laboratory experiment #2, shown here again in Fig.2. The system variables are

T external torque applied on rotor
θ angular position of rotor
ω angular velocity of rotor

The parameters of the system, shown in Fig.2, include