Week 3
Spring 2009

Lecture 5. Bayes estimation, minimaxity and Admissibility (cont.).

Admissibility

Conditions on priors and admissibility: conditions on the prior measure which guarantees that the corresponding generalized Bayes procedure is admissible.

Define

$$J_x(h) = \int h(\theta)\varphi(x-\theta)\,d\theta.$$

Let $S_1 = \{x \in \mathbb{R}^p : \|x\| \leq 1\}$.

Assumption

Growth Condition:

$$\int_{S_1^c} \frac{g(\theta)}{\|\theta\|^2 \log^2(\|\theta\|)} < \infty$$

Asymptotic flatness condition:

$$\int_{S_1^c} J_x\left\{\left\|\frac{\nabla g}{g} - \frac{J_x(\nabla g)}{J_x(g)}\right\|^2 g\right\} dx < \infty$$

It can be shown that $\int_{S_1^c} \|\nabla g\|^2 / g dx < \infty$ implies the flatness condition.

Theorem. Let G be a prior satisfying two conditions above. Then δ_G is admissible.

For the normal mean estimation problem with squared error loss,

Blyth's Method. Let δ be an estimator. Let $\{G_j\}$ be a sequence of finite prior measures such that:(i) $r(G_j, \delta) - r(G_j, \delta_{G_j}) \to 0$ as $j \to \infty$; (ii) $\inf_j \{G_j(S_1)\} > 0$. Then δ is an admissible estimator.

<u>*hint*</u>: Let $\delta'' = (\delta' + \delta)/2$. If $R(\theta, \delta') \leq R(\theta, \delta)$ for all θ and with strict inequality for some θ, then $R(\theta, \delta'') < R(\theta, \delta)$ for all θ. For all j,

$$\begin{aligned} r(G_j, \delta_{G_j}) &\leq r(G_j, \delta'') \leq \int_{S_1} R(\theta, \delta'') G_j(d\theta) + \int_{S_1^c} R(\theta, \delta) G_j(d\theta) \\ &= r(G_j, \delta) + \int_{S_1} \left[R(\theta, \delta'') - R(\theta, \delta)\right] G_j(d\theta) \leq r(G_j, \delta) - \varepsilon \end{aligned}$$

where $\varepsilon = \int_S \left[R(\theta, \delta'') - R(\theta, \delta)\right] G_j(d\theta)$. Contradiction!

Example. Let $X \sim N(\theta, 1)$. Let $g_j(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2j}\right)$, and $g_j = dG_j/d\mu$ where μ is the Lebesgue measure. It is easy to show

$$r(G_j, X) = \sqrt{j}, r(G_j, \delta_{G_j}) = \sqrt{j}\frac{j}{j+1}$$

then

$$r(G_j, X) = \frac{\sqrt{j}}{j+1} \to 0.$$

Proposition.

$$r(G, \delta) - r(G, \delta_G) = \int \|\delta_G - \delta\|^2 g^*(x)\, dx$$

<u>*Proof :*</u>

$$\begin{aligned}
r(G, \delta) - r(G, \delta_G) &= E_X E\left(\|\theta - \delta_G + \delta_G - \delta\|^2 - \|\theta - \delta_G\|^2 \,|X\right) \\
&= E_X E\left(\|\delta_G - \delta\|^2 \,|X\right) = \int \|\delta_G - \delta\|^2 g^*(x)\, dx
\end{aligned}$$

<u>*Proof of the theorem:*</u> Please read page 374 of Stein (1961).
Define $g_j = h_j^2 g$ where

$$h_j = \begin{cases} 1 & \|\theta\| \le 1 \\ 1 - \frac{\log(\|\theta\|)}{\log j} & 1 \le \|\theta\| \le j \\ 0 & \|\theta\| > j \end{cases}, \; j = 2, 3, \ldots.$$

It is easy to see

$$\delta_{G_j} = \frac{\int \theta h_j^2 g(\theta)\, \varphi(x - \theta)\, d\theta}{\int h_j^2 g(\theta)\, \varphi(x - \theta)\, d\theta} \to \delta_G \text{ a.s.}$$

and

$$g_j^*(x) = \int h_j^2 g(\theta)\, \varphi(x - \theta)\, d\theta \le g^*(x).$$

Write

$$\begin{aligned}
\delta_G(x) &= x + \frac{\nabla g^*}{g^*} = x + \frac{J_x(\nabla g)}{J_x(g)}, \\
\delta_{G_j}(x) &= x + \frac{\nabla g^*}{g^*} = x + \frac{J_x\left(h_j^2 \nabla g + g \nabla h_j^2\right)}{J_x\left(h_j^2 g\right)}
\end{aligned}$$

where the second equality for each equation follows from integration by parts.
Hence

$$\begin{aligned}
& r(G_j, \delta_G) - r(G_j, \delta_{G_j}) \\
=\; & \int_{S_1^c} \|\delta_G - \delta_{G_j}\|^2 g_j^*(x)\, dx + \int_{S_1^c} \|\delta_G - \delta_{G_j}\|^2 g_j^*(x)\, dx \;\; (\leftarrow \text{ apply DCT, since } g_j^*(x) \le g^*(x) \text{ finite.}) \\
\le\; & 2 \int_{S_1^c} \left\| \frac{J_x\left(g \nabla h_j^2\right)}{J_x(g_j)} \right\|^2 g_j^*(x)\, dx + 2 \int_{S_1^c} \left\| \frac{J_x(\nabla g)}{J_x(g)} - \frac{J_x\left(h_j^2 \nabla g\right)}{J_x(g_j)} \right\|^2 g_j^*(x)\, dx + o(1) \\
=\; & 2A_j + 2B_j + o(1)
\end{aligned}$$

Show $A_j \to 0$ by DCT:

$$\begin{aligned}
A_j &= \int_{S_1^*} \left\| \frac{J_x\left(g\nabla h_j^2\right)}{J_x\left(g_j\right)} \right\|^2 g_j^*\left(x\right) dx \\
&= 4\int_{S_1^*} \left\| \frac{J_x\left(gh_j\nabla h_j\right)}{J_x\left(g_j\right)} \right\|^2 g_j^*\left(x\right) dx = 4\int_{S_1^*} \left\| \frac{J_x\left(g_j^{1/2}\cdot g_j^{1/2}\nabla h_j\right)}{J_x\left(g_j\right)} \right\|^2 g_j^*\left(x\right) dx \\
&\leq 4\int_{S_1^*} J_x\left(g\left\|\nabla h_j\right\|^2\right) dx \text{ (Cauchy-Schwartz inequality and } g_j \leq g\text{)} \\
&\leq 4\int_{S_1^*} \left\|\nabla h_j\left(\theta\right)\right\|^2 g\left(\theta\right) d\theta
\end{aligned}$$

and

$$\left\|\nabla h_j\left(\theta\right)\right\|^2 = \frac{1}{\left\|\theta\right\|^2 \log^2\left(j\right)} I_{[1,j]}\left(\left\|\theta\right\|\right) \leq \frac{1}{\left\|\theta\right\|^2 \log^2\left(\left\|\theta\right\| \vee 2\right)} I_{[1,j]}\left(\left\|\theta\right\|\right).$$

Show $B_j \to 0$ by DCT again:

$$\begin{aligned}
&\left\| \frac{J_x\left(\nabla g\right)}{J_x\left(g\right)} - \frac{J_x\left(h_j^2\nabla g\right)}{J_x\left(g_j\right)} \right\|^2 g_j^*\left(x\right) \text{ (Note that } h_j^2\nabla g \text{ is } g_j\frac{\nabla g}{g}\text{)} \\
&= \frac{\left\| J_x\left(g_j\frac{J_x(\nabla g)}{J_x(g)} - h_j^2\nabla g\right)\right\|^2}{J_x\left(g_j\right)} \\
&= \frac{\left\| J_x\left[g_j\left(\frac{J_x(\nabla g)}{J_x(g)} - \frac{\nabla g}{g}\right)\right]\right\|^2}{J_x\left(g_j\right)} \\
&\leq J_x\left(g_j\left\| \frac{J_x\left(\nabla g\right)}{J_x\left(g\right)} - \frac{\nabla g}{g}\right\|^2\right) \text{ (Cauchy-Schwartz inequality)} \\
&\leq J_x\left(g\left\| \frac{J_x\left(\nabla g\right)}{J_x\left(g\right)} - \frac{\nabla g}{g}\right\|^2\right).
\end{aligned}$$

Admissibility of $\delta_0 = X$ for $p = 1, 2$

Let $g\left(\theta\right) = 1$, then

$$\int_{S_1^*} \frac{g\left(\theta\right)}{\left\|\theta\right\|^2 \log^2\left(\left\|\theta\right\|\right)} = 2\pi\int_2^\infty \frac{1}{r\log^2 r} dr < \infty.$$

Homework problem (you pick one part to work on). Let $X_i \sim \text{Poisson}\left(\lambda_i\right)$ be independent, $i = 1, 2, \ldots, p$. Denote $X = \left(X_1, \ldots, X_p\right)$ and $\lambda = \left(\lambda_1, \ldots, \lambda_p\right)$. Under the loss $L\left(\lambda, \delta\right) = \sum_{i=1}^p \left(\delta_i - \lambda_i\right)^2 / \lambda_i$, show that (i) for $p = 1$, X is an admissible estimator of λ using Blyth's method; (2) for $p \geq 2$, X is not an admissible estimator of λ.