CHAPTER 13

Vector Algebra

§13.1. Basic Concepts

A *vector* **V** in the plane or in space is an arrow: it is determined by its length, denoted $|\mathbf{V}|$ and its direction. Two arrows represent the same vector if they have the same length and are parallel (see figure 13.1). We use vectors to represent entities which are described by magnitude and direction. For example, a force applied at a point is a vector: it is completely determined by the magnitude of the force and the direction in which it is applied. An object moving in space has, at any given time, a direction of motion, and a speed. This is represented by the velocity vector of the motion. More precisely, the velocity vector at a point is an arrow of length the speed (ds/dt), which lies on the tangent line to the trajectory. The success and importance of vector algebra derives from the interplay between geometric interpretation and algebraic calculation. In these notes, we will define the relevant concepts geometrically, and let this lead us to the algebraic formulation.

Figure 13.1

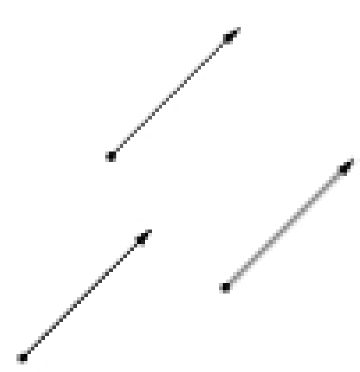
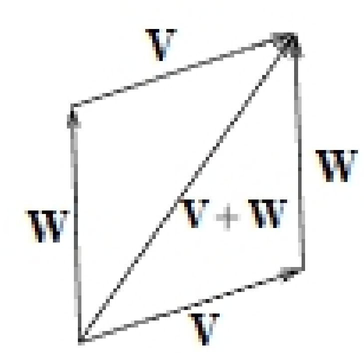


Figure 13.2



Newton did not write in terms of vectors, but through his diagrams we see that he clearly thought of forces in these terms. For example, he postulated that two forces acting simultaneously can be treated as acting sequentially. So suppose two forces, represented by vectors **V** and **W**, act on an object at a particular point. What the object feels is the *resultant* of these two forces, which can be calculated by placing the vectors end to end (as in figure 13.2). Then the resultant is the vector from the initial point of the first vector to the end point of the second. Clearly, this is the same if we reverse the order of the vectors. We call this the **sum** of the two vectors, denoted $\mathbf{V}+\mathbf{W}$. For example, if an object is moving in a fluid in space with a velocity **V**, while the fluid is moving with velocity **W**, then the object moves (relative to a fixed point) with velocity $\mathbf{V}+\mathbf{W}$.

Definition 13.1
a) A **vector** *represents the length and direction of a line segment. The* **length** *is denoted* $|\mathbf{V}|$. *A* **unit vector** $\mathbf{U}$ *is a vector of length 1. The* **direction** *of a vctor* $\mathbf{V}$ *is the unit vector* $\mathbf{U}$ *parallel to* $\mathbf{V}$: $\mathbf{U} = \mathbf{V}/|\mathbf{V}|$.
b) Given two points P, Q, the vector from P to Q is denoted $\vec{PQ}$.
c) Addition. The **sum**, *or* **resultant**, $\mathbf{V}+\mathbf{W}$ *of two vectors* $\mathbf{V}$ *and* $\mathbf{W}$ *is the diagonal of the parallelogram with sides* $\mathbf{V},\mathbf{W}$.
d) Scalar Multiplication. To distinguish them from vectors, real numbers are called **scalars**. *If c is a positve real number, $c\mathbf{V}$ is the vector with the same direction as $\mathbf{V}$ and of length $c|\mathbf{V}|$. If c negative, it is the same, but directed in the opposite direction.*

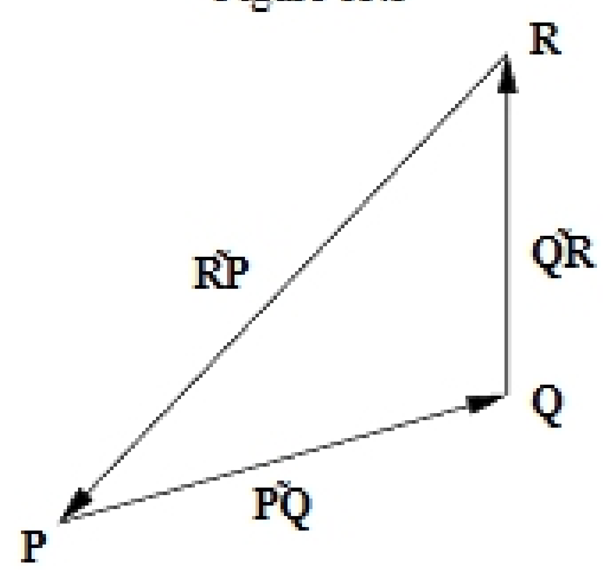
We note that the vectors $\mathbf{V}$, $c\mathbf{V}$ are parallel, and conversely, if two vectors are parallel (that is, they have the same direction), then one is a scalar multiple of the other.

Example 13.1 Let P, , Q, R be three points in the plane not lying on a line. Then

$$\vec{PQ} + \vec{QR} + \vec{RP} = \mathbf{0}\,. \tag{13.1}$$

From figure 13.3, we see that the vector $\vec{RP}$ is the same line segment as $\vec{PQ}+\vec{QR}$, but points in the opposite direction. Thus $\vec{RP} = -(\vec{PQ}+\vec{QR})$.

Figure 13.3



Example 13.2 Using vectors, show that if two triangles have corresponding sides parallel, that the lengths of corresponding sides are proportional.

Represent the sides of the two triangles by $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{U}'$, $\mathbf{V}'$, $\mathbf{W}'$ respectively. The hypothesis is that there are scalars a, b, c such that $\mathbf{U}' = a\mathbf{U}$, $\mathbf{V}' = b\mathbf{U}$, $\mathbf{W}' = c\mathbf{W}$. The conclusion is that $a = b = c$. To show this, we start with the result of example 1; since these are the sides of a triangle, we have

$$\mathbf{U}+\mathbf{V}+\mathbf{W} = \mathbf{0}\,, \quad \mathbf{U}'+\mathbf{V}'+\mathbf{W}' = \mathbf{0}\,, \quad \text{or, what is the same,} \quad a\mathbf{U}+b\mathbf{V}+c\mathbf{W} = \mathbf{0} \tag{13.2}$$

The first equation gives us $\mathbf{U} = -\mathbf{V}-\mathbf{W}$, which, when substituted in the last equation gives

$$(b-a)\mathbf{V} + (c-a)\mathbf{W} = \mathbf{0} \tag{13.3}$$

Now, if $b \neq a$, this tells us that $\mathbf{V}$ and $\mathbf{W}$ are parallel, and so the triangle lies on a line: that is, there is no triangle. Thus we must have $b = a$, and by the same reasoning, $c = a$ also.

§13.2. Vectors in the Plane

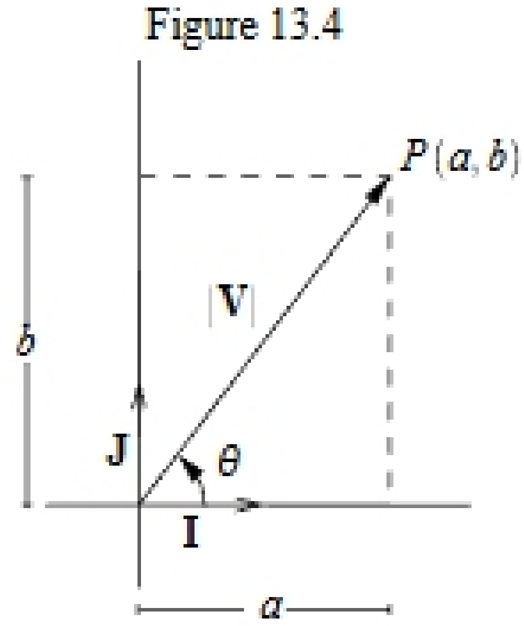
The advantage gained in using vectors is that they are moveable, and not tied to any particular coordinate system. As we have seen in the examples of the previous section, geometric facts can be easily derived using vectors while working in coordinates may be cumbersome. However, it is often the case, that in working with vectors we must do calculations in a particular coordinate system. It is important to realize that it is the worker who gets to choose the coordinates; it is not necessarily inherent in the problem.

We now explain how to move back and forth between vectors and coordinates. Suppose, then, that a coordinate system has been chosen: a point O, the origin, and two perpendicular lines through the origin, the x- and y-axes. A vector $\mathbf{V}$ is determined by its length, $|\mathbf{V}|$ and its direction, which we can describe by the angle θ that $\mathbf{V}$ makes with the horizontal (see figure 13.4). In this figure, we have realized $\mathbf{V}$ as the vector $\vec{OP}$ from the origin to P. Let (a,b) be the cartesian coordinates of P. Note that $\mathbf{V}$ can be realized as the sum of a vector of length a along the x-axis, and a vector of length b along the y-axis. We express this as follows.

Definition 13.2 *We let* $\mathbf{I}$ *represent the vector from the origin to the point (1,0), and* $\mathbf{J}$ *the vector from the origin to the point (0,1). These are the* **basic** *unit vectors (a unit vector is a vector of length 1). The unit vector in the direction* θ *is* $\cos\theta\mathbf{I}+\sin\theta\mathbf{J}$.

If $\mathbf{V}$ is a vector of length r and angle θ, then $\mathbf{V}=r(\cos\theta\mathbf{I}+\cos\theta\mathbf{J})$. If $\mathbf{V}$ is the vector from the origin to the point (a,b); r is the length of $\mathbf{V}$, and $\cos\theta\mathbf{I}+\cos\theta\mathbf{J}$ is its direction. If $P(a,b)$ is the endpoint of $\mathbf{V}$, then $\mathbf{V}=\vec{OP}=a\mathbf{I}+b\mathbf{J}$. a and b are called the **components** of $\mathbf{V}$.

Figure 13.4



Of course, r and θ are the usual polar coordinates, and we have these relations:

$$|\mathbf{V}|=\sqrt{a^2+b^2}\,,\quad \theta=\arctan\frac{b}{a}\,,\qquad a=|\mathbf{V}|\cos\theta\,,\quad b=|\mathbf{V}|\sin\theta\,. \tag{13.4}$$

We add vectors by adding their components, and multiply a vector by a scalar by multiplying the components by the scalar.

Proposition 13.2 *If* $\mathbf{V}=a\mathbf{I}+b\mathbf{J}$ *and* $\mathbf{W}=c\mathbf{I}+d\mathbf{J}$, *then* $\mathbf{V}+\mathbf{W}=(a+c)\mathbf{I}+(b+d)\mathbf{J}$.

This is verified in figure 13.5.