

On Image Analysis by the Methods of Moments

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Abstract—Various types of moments have been used to recognize image patterns in a number of applications. This paper evaluates a number of moments and addresses some fundamental questions, such as image representation ability, noise sensitivity, and information redundancy. Moments considered here include regular moments, Legendre moments, Zernike moments, pseudo-Zernike moments, rotational moments, and complex moments. Properties of these moments are examined in detail and the interrelationships among them are discussed. Both theoretical and experimental results are presented.

Index Terms—Image analysis, image reconstruction, image representation, invariance, moment invariants, moments, pattern recognition.

I. INTRODUCTION

MOMENTS and functions of moments have been utilized as pattern features in a number of applications to achieve invariant recognition of two-dimensional image patterns [1]–[9]. Hu [1] first introduced moment invariants in 1961, based on methods of algebraic invariants. Using nonlinear combinations of *regular moments* (regular moments will be referred to as *geometric moments*), he derived a set of invariant moments which has the desirable properties of being invariant under image translation, scaling, and rotation. However, the question of what is gained by including higher order moments in the context of image analysis has not been addressed, and the recovery of the image from these moments is deemed to be quite difficult.

Teague [10] has suggested the notion of orthogonal moments to recover the image from moments based on the theory of orthogonal polynomials, and has introduced *Zernike moments*, which allow independent moment invariants to be constructed easily to an arbitrarily high order. Other orthogonal moments are *Legendre moments*, making use of Legendre polynomials. In [11], *rotational moments* are used to extend the definition of moment invariants to arbitrary order in a manner which ensures that their magnitudes do not diminish significantly with increasing order. More recently, the notion of *complex moments* [12], [13] has been introduced as a simple and straightforward way to derive moment invariants. The definitions of these various types of moments and a summary of their properties are presented in Section II.

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From the point of view of pattern recognition, moment invariants are considered reliable features if their values are insensitive to the presence of image noise. The effects of sampling, digitizing, and quantization noise on moment invariants have been addressed in [14]. In Section III of this paper, the effect of image noise on various types of moments is analyzed using stochastic images. Second-order statistics of various moments are related to those of the image noise so that the impact of noise can be estimated and compared. In Section IV, information redundancy among different orders of moments is derived for various moment types as an indication of their information content.

It is well known that, in principle, most of the image information can be recaptured by using a sufficiently large number of a particular set of image moments. In Section V, the question of how well an image can be characterized by a small finite set of its moments is investigated by first reconstructing the image from its moments and then evaluating the mean-square reconstruction error. Theoretical results for stochastic images under both noise-free and noisy conditions are derived. The error is expressed as a function of the statistical properties of the image function and the noise process. Experimental results for a set of chosen test images are also presented.

II. MOMENTS

In this section, the various types of moments are defined and their properties briefly summarized. We assume that the real image intensity function $f(x, y)$ is piecewise continuous and has bounded support.

A. Geometric Moments (GM)

The geometric moments of order $(p + q)$ of $f(x, y)$ are defined as

$$M_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q f(x, y) dx dy \quad (2.1)$$

where $p, q = 0, 1, 2, \dots, \infty$. The above definition has the form of the projection of the function $f(x, y)$ onto the monomial $x^p y^q$. However, the basis set $\{x^p y^q\}$, while complete (Weierstrass approximation theorem [15]), is not orthogonal.

B. Legendre Moments (LM)

The Legendre moments of order $(m + n)$ are defined as

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_m(x) P_n(y) f(x, y) dx dy \quad (2.2)$$

where $m, n = 0, 1, 2, \dots, \infty$. The Legendre polynomials $\{P_m(x)\}$ [15] are a complete orthogonal basis set on the interval $[-1, 1]$:

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2m+1} \delta_{mn} \quad (2.3)$$

The n th-order Legendre polynomial is

$$P_n(x) = \sum_{j=0}^n a_{nj} x^j = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (2.4)$$

By the orthogonality principle, the image function $f(x, y)$ can be written as an infinite series expansion in terms of the Legendre polynomials over the square $[-1 \leq x, y \leq 1]$:

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} P_m(x) P_n(y) \quad (2.5)$$

where the Legendre moments $\{\lambda_{mn}\}$ are computed over the same square. If only Legendre moments of order $\leq N$ are given, then the function $f(x, y)$ can be approximated by a continuous function which is a truncated series:

$$f(x, y) \approx \sum_{m=0}^N \sum_{n=0}^m \lambda_{m-n, n} P_{m-n}(x) P_n(y) \quad (2.6)$$

Using (2.1), (2.2), and (2.4), the Legendre moments and geometric moments are related by

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{4} \sum_{j=0}^m \sum_{k=0}^n a_{mj} a_{nk} M_{jk} \quad (2.7)$$

Thus, a given Legendre moment depends only on geometric moments of the same order and lower, and conversely.

C. Zernike Moments (ZM)

The complex Zernike moments of order n with repetition l are defined as

$$A_{nl} = \frac{n+1}{\pi} \int_0^{2\pi} \int_0^{\infty} [V_{nl}(r, \theta)]^* \cdot f(r \cos \theta, r \sin \theta) r dr d\theta \quad (2.8)$$

where $n = 0, 1, 2, \dots, \infty$ and l takes on positive and negative integer values subject to the conditions

$$n - |l| = \text{even}, \quad |l| \leq n \quad (2.9)$$

The symbol $*$ denotes the complex conjugate. The Zernike polynomials [16]

$$v_{nl}(x, y) = \int_0^{2\pi} \int_0^{\infty} V_{nl}(r, \theta) = R_{nl}(r) e^{il\theta} \quad (2.10)$$

are a complete set of complex-valued functions orthogonal on the unit disk $x^2 + y^2 \leq 1$:

$$\int_0^{2\pi} \int_0^1 [V_{nl}(r, \theta)]^* V_{mk}(r, \theta) r dr d\theta = \frac{\pi}{n+1} \delta_{nm} \delta_{kl} \quad (2.11)$$

The real-valued radial polynomials $\{R_{nl}(r)\}$ satisfy the relations

$$\int_0^1 R_{nl}(r) R_{ml}(r) r dr = \frac{1}{2(n+1)} \delta_{nm} \quad (2.12)$$

and are defined as

$$R_{nl}(r) = \sum_{s=0}^{(n-|l|)/2} (-1)^s \frac{(n-s)!}{s! \left(\frac{n+|l|}{2} - s\right)! \left(\frac{n-|l|}{2} - s\right)!} r^{n-2s} = \sum_{\substack{k=|l| \\ n-k=\text{even}}}^n B_{n|l|k} r^k \quad (2.13)$$

The function $f(x, y)$ can be expanded in terms of the Zernike polynomials over the unit disk as

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{\substack{l=-\infty \\ n-|l|=\text{even} \\ |l| \leq n}}^{\infty} A_{nl} V_{nl}(x, y) \quad (2.14)$$

where the Zernike moments $\{A_{nl}\}$ are computed over the unit disk. If the series expansion is truncated at a finite order N , then the truncated expansion is the optimum approximation to $f(x, y)$:

$$f(x, y) \approx \sum_{n=0}^N \sum_{\substack{l=-\infty \\ n-|l|=\text{even} \\ |l| \leq n}}^{\infty} A_{nl} V_{nl}(x, y) \quad (2.15)$$

Note that because of the orthogonality of the Legendre and Zernike polynomials, both the Legendre moments $\{\lambda_{mn}\}$ in (2.5) and the Zernike moments $\{A_{nl}\}$ in (2.14) are in each case independent. It can be shown that Zernike moments and geometric moments are related by

$$A_{nl} = \frac{n+1}{\pi} \sum_{\substack{k=|l| \\ n-k=\text{even}}}^n \sum_{j=0}^q \sum_{m=0}^{|l|} w^m \cdot \binom{q}{j} \binom{|l|}{m} B_{n|l|k} M_{k-2j-m, 2j+m} \quad (2.16)$$

where

$$w = \begin{cases} -i & l > 0 \\ +i & l \leq 0, \end{cases} \quad (2.17)$$

$q = \frac{1}{2}(k - |l|)$, and $i = \sqrt{-1}$.

D. Pseudo-Zernike Moments (PM)

Zernike polynomials were first introduced in 1934 [17] and were later derived from the requirement of orthogo-

nality and invariance properties by Bhatia and Wolf [18]. Zernike polynomials, being invariant in form with respect to rotations of axis about the origin, are polynomials in x and y . A related orthogonal set of polynomials in x , y , and r was derived in [18] which has properties analogous to those of Zernike polynomials. This set of polynomials, which we shall call pseudo-Zernike polynomials, differs from that of Zernike in that the real-valued radial polynomials are defined as

$$R_{nl}(r) = \sum_{s=0}^{n-|l|} (-1)^s \frac{(2n+1-s)!}{s!(n-|l|-s)!(n+|l|+1-s)!} r^{n-s} \\ = \sum_{k=|l|}^n S_{n|l|k} r^k \quad (2.18)$$

where now $n = 0, 1, 2, \dots, \infty$ and l takes on positive and negative integer values subject to $|l| \leq n$ only. By simple enumeration, this set of pseudo-Zernike polynomials contains $(n+1)^2$ linearly independent polynomials of degree $\leq n$, whereas the set of Zernike polynomials contains only $\frac{1}{2}(n+1)(n+2)$ linearly independent polynomials of degree $\leq n$ due to the additional condition of $n - |l| = \text{even}$.

The Zernike moments in (2.8) become pseudo-Zernike moments if the radial polynomials $\{R_{nl}(r)\}$ in (2.18), which also satisfy the relations in (2.12), are used to compute the polynomials with the condition $n - |l| = \text{even}$ eliminated. Since the pseudo-Zernike polynomials are also a complete set of functions orthogonal on the unit disk, both the series expansions of $f(x, y)$ in (2.14) and (2.15) hold with the condition $n - |l| = \text{even}$ eliminated, and $\{A_{nl}\}$ and $\{V_{nl}(x, y)\}$ are now the pseudo-Zernike moments and the pseudo-Zernike polynomials, respectively. In addition, the pseudo-Zernike moments are independent. It is shown in Section III that pseudo-Zernike moments are less sensitive to image noise than are the conventional Zernike moments.

E. Rotational Moments (RM)

The rotational moments of order n with repetition l are defined as

$$D_{nl} = \int_0^{2\pi} \int_0^{\infty} r^n e^{-i\ell\theta} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (2.19)$$

where $n = 0, 1, 2, \dots, \infty$ and l takes on any positive and negative integer values. From (2.8), (2.10), (2.13), and (2.19), it can be shown that Zernike moments and rotational moments are related by

$$A_{nl} = \frac{n+1}{\pi} \sum_{\substack{k=|l| \\ n-k=\text{even}}}^n B_{n|l|k} D_{kl} \quad (2.20)$$

from which it follows that rotational moments can also be obtained from geometric moments by

$$D_{nl} = \sum_{j=0}^q \sum_{m=0}^{|l|} w^m \binom{q}{j} \binom{|l|}{m} M_{n-2j-m, 2j+m} \quad (2.21)$$

where w is given by (2.17) and $q = \frac{1}{2}(n - |l|)$. Unlike Legendre and Zernike moments, the rotational moments $\{D_{nl}\}$ are not generally independent.

F. Complex Moments (CM)

The notion of complex moments was recently introduced in [12] as a simple and straightforward way to derive moment invariants. The complex moments of order $(p+q)$ are defined as

$$C_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x, y) dx dy \quad (2.22)$$

where $p, q = 0, 1, 2, \dots, \infty$, and $i = \sqrt{-1}$. The complex moment of order $(p+q)$ is a linear combination with complex coefficients of the geometric moments $\{M_{rs}\}$ satisfying $r+s = p+q$:

$$C_{pq} = \sum_{r=0}^p \sum_{s=0}^q \binom{p}{r} \binom{q}{s} i^{p+q-(r+s)} \cdot (-1)^{q-s} M_{r+s, p+q-(r+s)} \quad (2.23)$$

In polar coordinates, the complex moment of order $(p+q)$ can be written as

$$C_{pq} = \int_0^{2\pi} \int_0^{\infty} r^{p+q} e^{i(p-q)\theta} f(r \cos \theta, r \sin \theta) r dr d\theta; \quad (2.24)$$

thus, it is related to the rotational moments as

$$C_{pq} = D_{n=1/2(n-1), 1/2(n+1)} = C_{pq} \quad (2.25)$$

from which it also follows that Zernike moments and complex moments are related by

$$A_{nl} = \frac{n+1}{\pi} \sum_{\substack{k=|l| \\ n-k=\text{even}}}^n B_{n|l|k} C_{1/2(k-1), 1/2(k+1)} \quad (2.26)$$

From (2.24), the repetition of C_{pq} is defined as $p - q$. Like the rotational moments, the complex moments $\{C_{pq}\}$ are not independent.

III. NOISE SENSITIVITY

Moment invariants using various schemes [2], [8]-[11] based on the different moment types as defined in the previous section have been shown to provide perfect invariance properties under noise-free condition. However, in the presence of noise, the computed invariant moments are expected not to be strictly invariant. Thus, it is important to investigate which invariance scheme is less vulnerable to image noise. In this section, the various types of moments, and hence their corresponding moment invariants, of noisy images are analyzed and compared. Theoretical expressions relating the second-order statis-