Chapter 11 – Torque and Angular Momentum

I. Torque

II. Angular momentum
- Definition

III. Newton's second law in angular form

IV. Angular momentum
- System of particles
- Rigid body
- Conservation

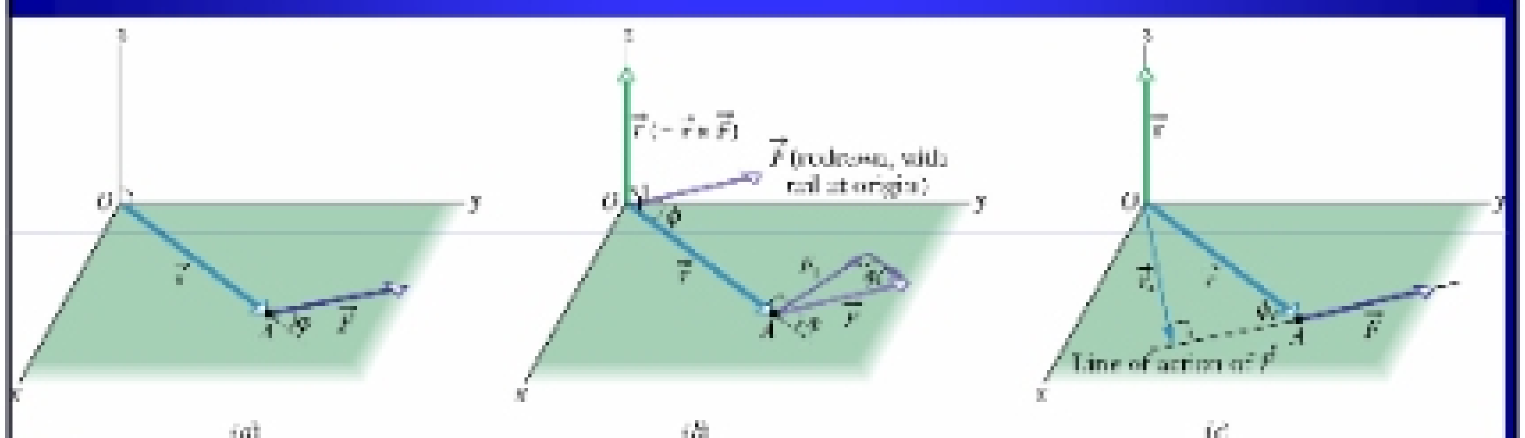
I. Torque

- Vector quantity. $\vec{\tau} = \vec{r} \times \vec{F}$

Direction: right hand rule.

Magnitude: $\tau = r \cdot F \sin\varphi = r \cdot F_{\perp} = (r \sin\varphi)F = r_{\perp}F$

Torque is calculated with respect to (about) a point. Changing the point can change the torque's magnitude and direction.



II. Angular momentum

- Vector quantity. $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$ Units: kg m²/s

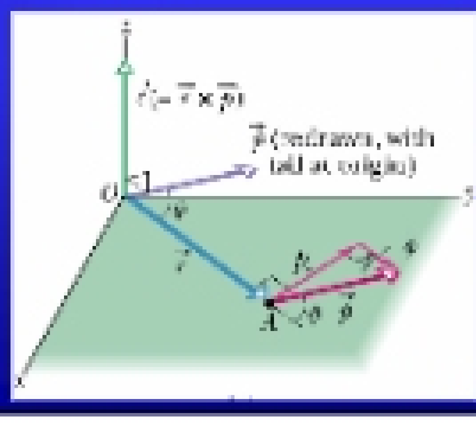
Magnitude: $l = r \cdot p \sin\varphi = r \cdot m \cdot v \sin\varphi = r \cdot m \cdot v_{\perp} = r \cdot p_{\perp} = (r \sin\varphi) p = r_{\perp} p = r_{\perp} m \cdot v$

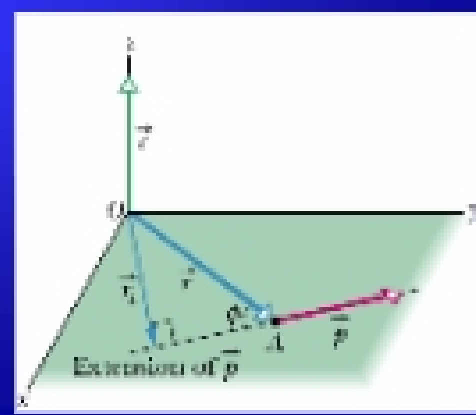
Direction: right hand rule.

$\vec{l}$ positive → counterclockwise

$\vec{l}$ negative → clockwise

Direction of $\vec{l}$ is always perpendicular to plane formed by $\vec{r}$ and $\vec{p}$.





III. Newton's second law in angular form

Linear	Angular	
$\vec{F}_{net} = \frac{d\vec{p}}{dt}$	$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$	Single particle

The vector sum of all torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Proof:

$$\vec{l} = m(\vec{r} \times \vec{v}) \rightarrow \frac{d\vec{l}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right) = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}) = m(\vec{r} \times \vec{a}) =$$

$$\frac{d\vec{l}}{dt} = \vec{r} \times m\vec{a} = \vec{r} \times \vec{F}_{net} = \sum(\vec{r} \times \vec{F}) = \vec{\tau}_{net}$$

V. Angular momentum

- System of particles: $L = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + ... + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$

$$\frac{d\vec{L}}{dt}=\sum_{i=1}^{n}\frac{d\vec{l}_i}{dt}=\sum_{i=1}^{n}\vec{\tau}_{net,i}\rightarrow\vec{\tau}_{net}=\frac{d\vec{L}}{dt}$$

Includes internal torques (due to forces between particles within system) and external torques (due to forces on the particles from bodies outside system).

Forces inside system → third law force pairs → torque_{int} sum =0 → The only torques that can change the angular momentum of a system are the external torques acting on a system.

The net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

- **Rigid body** (rotating about a fixed axis with constant angular speed ω):

Magnitude $l_i=(r_i)(p_i)(\sin 90^{\circ})=(r_i)(m_i v_i)$

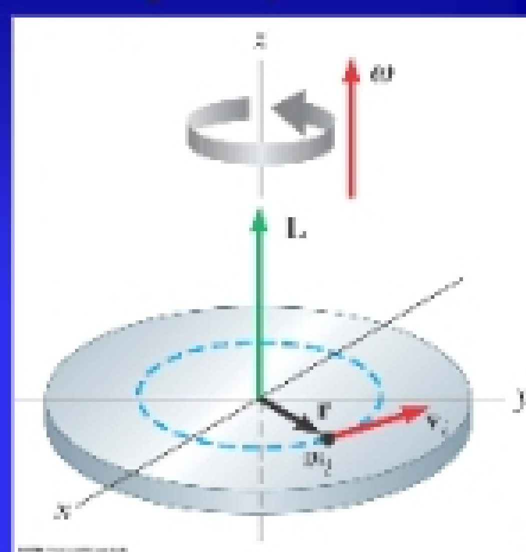
$v_i=\omega\cdot r_i$

$l_i=r_i m_i(\omega\, r_i)=\omega\, m_i r_i^2$

Direction: $\vec{l}_i$ → perpendicular to $\vec{r}_i$ and $\vec{p}_i$

$$L_z=\sum_{i=1}^{n}l_{iz}=\sum_{i=1}^{n}m_i r_i^2\omega=\left(\sum_{i=1}^{n}m_i\cdot r_i^2\right)\omega=I\omega$$
$$L_z=\omega\, I$$

$$\frac{dL_z}{dt}=I\frac{d\omega}{dt}=I\alpha\rightarrow\frac{dL_z}{dt}=\tau_{ext}$$



$$L=I\omega$$

Rotational inertia of a rigid body about a fixed axis