MA123, Chapter 10: Formulas for integrals: integrals, antiderivatives, and the Fundamental Theorem of Calculus (pp. 207-233, Gootman)

`Chapter Goals:`

- Understand the statement of the Fundamental Theorem of Calculus.
- Learn how to compute the antiderivative of some basic functions.
- Learn how to use the substitution method to compute the antiderivative of more complex functions.
- Learn how to solve area and distance traveled problems by means of antiderivatives.

`Assignments:` `Assignment 22` `Assignment 23` `Assignment 24 (Review)`

So far we have learned about the idea of the integral, and what is meant by computing the definite integral of a function $f(x)$ over the interval $[a, b]$. As in the case of derivatives, we now study procedures for computing the definite integral of a function $f(x)$ over the interval $[a, b]$ that are easier than computing limits of Riemann sums. As with derivatives, however, the definition is important because it is only through the definition that we can understand why the integral gives the answers to particular problems.

▶ Idea of the Fundamental Theorem of Calculus:

The easiest procedure for computing definite integrals is not by computing a limit of a Riemann sum, but by relating integrals to (anti)derivatives. This relationship is so important in Calculus that the theorem that describes the relationships is called the Fundamental Theorem of Calculus.

▶ Computing some antiderivatives:

In previous chapters we were given a function $f(x)$ and we found the derivative $f'(x)$. In this section, we will do the reverse. We will be given a function $f(x)$ that is the derivative of another function $F(x)$ and will compute $F(x)$. In other words find a function $F(x)$ such that $F'(x) = f(x)$. $F(x)$ is called an **antiderivative** of $f(x)$. For example $(x^3)' = 3x^2$ so an antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$.

Note that $F(x) = x^3 + 2$ is also an antiderivative of $f(x) = 3x^2$ because $(x^3 + 2)' = 3x^2$. In general, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ where C is any constant. This leads to the following notation.

Definition of the indefinite integral:

The *indefinite integral* of $f(x)$, denoted by

$$\int f(x)\,dx$$

without limits of integration, is the *general* antiderivative of $f(x)$.

For example, it is easy to check that $\int 3t^2\,dt = t^3 + c$, where c is any constant.

Recall that the power rule for derivatives gives us $(x^n)' = nx^{n-1}$. We multiply by n and subtract 1 from the exponent. Since antiderivatives are the reverse of derivatives, to compute an the antiderivative we first increase the power by 1, then divide by the new power.

The formulas below can be verified by differentiating the righthand side of each expression.

Some basic indefinite integrals:

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$

2. $\displaystyle\int e^x\,dx = e^x + C$

$n = -1$ in formula **1** leads to division by zero, but for this special case we may use $(\ln(x))' = \dfrac{1}{x}$:

3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

Rules for indefinite integrals:

A. $\displaystyle\int c\,f(x)\,dx = c\int f(x)\,dx$

B. $\displaystyle\int (f(x) \pm g(x))\,dx = \left(\int f(x)\,dx\right) \pm \left(\int g(x)\,dx\right)$

Example 1: Evaluate the indefinite integral $\displaystyle\int (t^3 + 3t^2 + 4t + 9)\,dt$.

Example 2: Evaluate the indefinite integral $\displaystyle\int \frac{6}{\sqrt{t}}\,dt$.

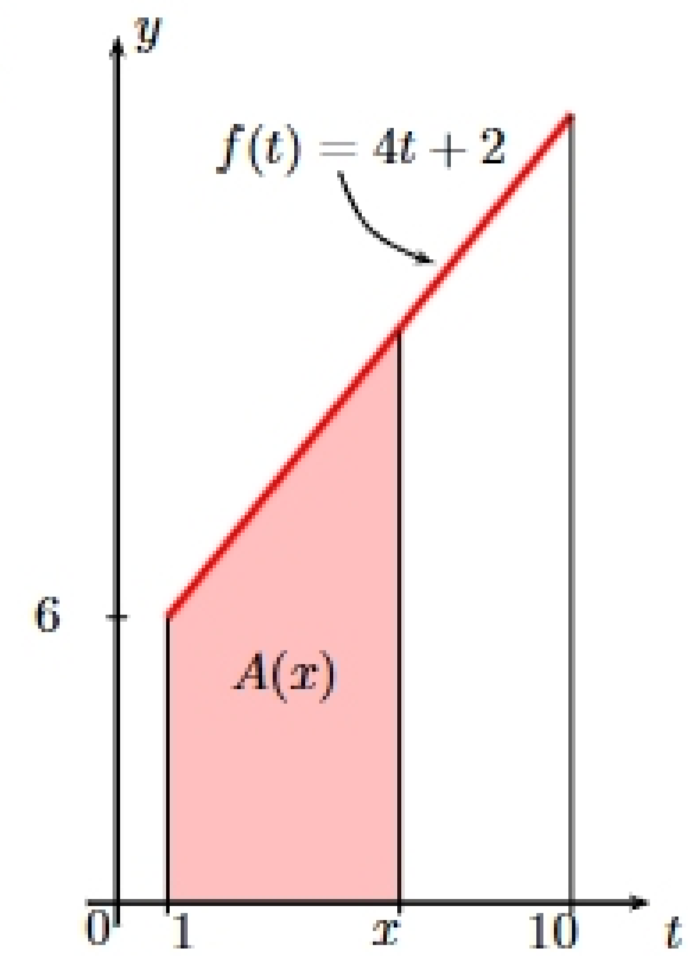
Warning: We do not have simple derivative rules for products and quotients, so we should not expect simple integral rules for products and quotients.

Example 3: Evaluate the indefinite integral $\displaystyle\int t^3(t+2)\,dt$.

Example 4: Evaluate the indefinite integral $\int \frac{x^2+9}{x^2}\,dx$.

We now have some experience computing antiderivatives. We will now see how antiderivatives give us an elegant method for finding areas under curves.

Example 5: Find a formula for $A(x) = \int_1^x (4t+2)\,dt$, that is, evaluate the definite integral of the function $f(t) = 4t+2$ over the interval $[1, x]$ inside $[1, 10]$. (**Hint:** think of this definite integral as an area.) Find the values $A(5), A(10), A(1)$. What is the derivative of $A(x)$ with respect to x?



Observations: There are two important things to notice about the function $A(x)$ analyzed in Example 1:

$$A(1) = \int_1^1 (4t+2)\,dt = 0 \qquad\qquad A'(x) = \frac{d}{dx}\Bigg(\underbrace{\int_1^x (4t+2)\,dt}_{A(x)}\Bigg) = 4x+2.$$

Notice what the last equality says: The instantaneous rate of change of the area under the curve $y = 4t+2$ at $t = x$ is simply equal to the value of the curve evaluated at $t = x$.

Why? $A(x)$ measures the area of some geometric figure. As x increases, the width of the figure increases, and so the area increases. $A'(x)$ measures the rate of increase of the figure. Now, as x increases, the right wall of the figure sweeps out additional area, so the rate at which the area increases should be equal to the height of the right wall.

The following pages will make this idea more precise.