

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #6 2010

Assignments handed in by 6 PM on Thursday Oct. 21 will receive a 5 point bonus. Assignments handed in after that but by 4 PM on Friday Oct. 22 will receive full credit but no bonus. No assignments will be accepted after 4 PM on Oct. 22.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
3. Separation of Variables, Part I		
3.6 Beyond Fourier Series	Th Oct 14	---
4. Sturm Liouville Theory		
4.1 Eigenfunctions and Eigenvalues	F, M Oct 15, 18	5.1-5.3, 5.5
4.2 Eigenfunction Expansions	W Oct 20	5.3

PROBLEMS

(1) (60 points) Consider the regular Sturm-Liouville problem given below.

$$\frac{d}{dx} \left(x^3 \frac{dF}{dx} \right) = -\lambda x^3 F, \quad 1 < x < 3, \quad \text{with } F(1) = 0 \text{ and } F(3) = 0.$$

(a) (10 points) Prove that the eigenvalues are positive.

(b) (10 points) From your work in MTH 163 or MTH 165, you should recognize the equation as an equidimensional equation. Such equations have solutions of the form x^r , where the possible values of r are determined by substituting the form into the equation. Recall also that for any real constant α , $x^{i\alpha} = \cos(\alpha \ln[x]) + i \sin(\alpha \ln[x])$. Use this information to find the general solution. In doing this, you may assume for now that $\lambda > 4$. This will be explored further in part (f) of this problem. Show that the eigenvalues and eigenfunctions of this system are given by

$$\lambda_n = 4 + \frac{(n\pi)^2}{(\ln(3))^2}, \quad F_n(x) = \frac{1}{x^2} \sin\left(\frac{n\pi \ln(x)}{\ln(3)}\right), \quad n = 1, 2, 3, \dots$$

(c) (10 points) Verify that the eigenfunctions satisfy the appropriate orthogonality relation.

(d) (10 points) Use Mathematica to plot the first five eigenfunctions. Verify for these five that the n^{th} eigenfunction has $n-1$ zeros in the interior of the interval.

(e) (10 points) Obtain explicitly the expansion in these eigenfunctions of $f(x) = 1/x^2$.

Use Mathematica to plot the Nth partial sum of your series for an appropriate value of N, and show on the same plot the function $f(x)$.

(f) (10 points) Prove that $\lambda > 4$. (Hint: A direct attack based on solving the equation works.)

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(2) (40 points) Consider the boundary value problem for $\Phi(x,t)$ given below.

$$x^3 \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial x} \left(x^3 \frac{\partial \Phi}{\partial x} \right), \quad 1 < x < 3 \text{ and } t > 0,$$

with $\Phi(1,t) = 0$, $\Phi(3,t) = 0$, and $\Phi(x,0) = \frac{1}{x^2}$.

(a) (30 points) Use your results of problem (1) and separation of variables to solve the problem.

(b) (10 points) Find a simple one-term approximation for large times. Estimate the time range for which your approximation is valid.

CHALLENGE PROBLEM

In this problem you will explore the use of the Rayleigh quotient to estimate the lowest eigenvalue of Sturm-Liouville system. We will start with a brief review. The general Sturm-Liouville equation has the form

$$Ly = -\lambda py, \text{ where } L = \frac{d}{dx} \left[r(x) \frac{d}{dx} \right] - q(x).$$

Here $r(x)$ must be positive and have a continuous derivative, $p(x)$ must be positive and continuous, and $q(x)$ must be continuous. To keep the calculations as straightforward as possible, we consider the special boundary conditions

$$y(a) = 0 \text{ and } y(b) = 0.$$

By multiplying the equation by y and integrating, we get

$$\lambda = - \frac{\int_a^b yLy \, dx}{\int_a^b py^2 \, dx} = \frac{\int_a^b (ry'^2 + qy^2) \, dx}{\int_a^b py^2 \, dx}.$$

The first form follows directly, and the second form is established with an integration-by-parts and use of the boundary conditions. The second form is one we used in class to prove that the eigenvalues are positive if $q \geq 0$. It is the first form that will be more convenient in this problem. We now make a significant shift of our point-of-view on the above expression. Let S be the set of real functions $y(x)$ which are twice continuously differentiable and which satisfy the zero boundary conditions at a and b . Then we define the functional $F[y]$ on S as follows:

$$F[y] = - \frac{\int_a^b yLy \, dx}{\int_a^b py^2 \, dx}.$$

If $y_n(x)$ is the n th eigenfunction of the Sturm-Liouville system, the value of F is λ_n , the n th eigenvalue. In this problem we are going to turn this around, and try to use the expression $F[y]$ to estimate the first eigenvalue of the Sturm-Liouville system. This would seem to be difficult if

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CHALLENGE PROBLEM (continued) we don't know the eigenfunction, and of course if we do know the eigenfunction we already know the eigenvalue.

Before actually beginning the problem, we also review briefly the expansion theorem associated with the Sturm-Liouville system. Given a piecewise smooth $f(x)$, the associated Sturm-Liouville expansion is

$$\sum_{n=1}^{\infty} C_n y_n(x), \text{ where } C_n = \frac{\int_a^b f(x)p(x)y_n(x)dx}{N_n}, \text{ and where } N_n \text{ is } \int_a^b p(x)y_n^2(x)dx.$$

This converges everywhere to $\frac{1}{2}[f(x-) + f(x+)]$.

(a) (25 points) Given any y in S , express y in terms of the y_n 's : $y = \sum_{n=1}^{\infty} C_n y_n$. Now show that

$$F[y] = \frac{\sum_{n=1}^{\infty} \lambda_n C_n^2 N_n}{\sum_{n=1}^{\infty} C_n^2 N_n}.$$

(b) (25 points) Use your result of part (a) to show that for any y in S , $F[y] \geq \lambda_1$. Thus the functional F evaluated for any y in S , gives an upper bound on the first eigenvalue. If we have reason to believe that our trial y is close in shape to the true first eigenfunction, then we have reason to hope that our upper bound is reasonably close to the eigenvalue. In choosing a trial y with which to estimate the first eigenvalue, we should use what we know about the first eigenfunction – namely, that it has no interior zeros in the interval $[a,b]$.

(c) (25 points) In these last two parts of the problem, you will apply your results above to the simple Sturm-Liouville system given below.

$$y'' + \lambda y = 0, \quad 0 < x < 1, \text{ with } y(0) = 0 \text{ and } y(1) = 0.$$

As we showed in class many times, the first eigenvalue here is $\lambda_1 = \pi^2 = 9.8696$. Use the trial function $y(x) = x(1-x)$ to estimate this eigenvalue.

(d) (25 points) Because the estimate is always an upper bound, there are rational ways to improve it. We choose a trial function $y(x) = x(1-x) + ax^2(1-x)^2$. Now the answer will depend on a , and for any a it will still be an upper bound. To get the best estimate with a trial function of this form, we minimize the answer with respect to a . Carry this out for the system of part (c).

(e) **Bonus Question** (honor but no points) Consider the functional $\lambda = \frac{\int_a^b (ry'^2 + qy^2)dx}{\int_a^b py^2 dx}$.

Using the techniques of the calculus of variations, show that the variational equation of this functional, obtained under the conditions $y(a) = 0$ and $y(b) = 0$, is the original Sturm-Liouville equation.