

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #8 2011

Assignments handed in by 6 PM on Wednesday Nov. 2 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday Nov. 3 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Nov. 3. This is the last homework assignment before Exam #2 on Nov. 10.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
V. SEPARATION OF VARIABLES, PART 2		
5.4 Organ Pipe Acoustics	W Oct 26	---
5.5 Standing Acoustic Modes in Three Dimensions	Th Oct 27	7.3,7.4
VI. UNBOUNDED DOMAINS		
6.1 Fourier Integral	F Oct 28	10.1-10.3
6.2 Fourier Transform	M Oct 31	10.3,10.4

PROBLEMS

(1) (30 points) A flexible beam occupies the region $0 \leq x \leq L$. The ends at $x = 0$ and $x = L$ are simply supported, which means that the ends are fixed in position, but can rotate freely in response to bending moments. If the beam is displaced from equilibrium, it will vibrate, somewhat like a stretched string displaced from equilibrium. The vertical displacement y is a function of both position x , and time t . In solid mechanics (e.g. in ME 226) it is shown that the equation and boundary conditions governing the vibrations are

$$\frac{\partial^2 y}{\partial t^2} = -\sigma \frac{\partial^4 y}{\partial x^4}, 0 < x < L \text{ and } t > 0,$$

$$\text{with } y(0,t) = 0, \frac{\partial^2 y}{\partial x^2}(0,t) = 0, y(L,t) = 0, \text{ and } \frac{\partial^2 y}{\partial x^2}(L,t) = 0.$$

The parameter σ is a combination of material properties and geometric properties: $\sigma = \frac{EI}{\rho A}$.

Here E is the Young's modulus (relating stress and strain), ρ is the material density, A is the cross-sectional area of the beam, and I is a geometric property called the moment of inertia.

(a) (15 points) Find the frequencies of the normal modes of the beam. It is possible to do this by separation of variables, but it is somewhat lengthy. That method will be presented in the solution sheet. Here it is suggested that you try to guess the mode shapes by considering some very familiar functions that are eigenfunctions of $\partial^2 / \partial x^2$ and satisfy both boundary conditions at both ends of the beam.

(b) (15 points) Consider a beam with a rectangular cross section with horizontal dimension $w = 3$ cm and vertical dimension $b = 1$ mm. For such a section, it can be shown that $I = wb^3 / 12$. The length of the beam is $L = 1$ m. The Young's modulus is $E = 2.07 \times 10^{11}$ N/m². An observation of the fundamental mode shows that the linear frequency of that mode is 2.33 Hz. What is the density ρ of the material of the beam? What is the linear frequency of the second mode of vibration?

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(2) (25 points)) In this problem you will explore several basic properties of eigenfunctions of the Laplace operator in three dimensions. Let V be a volume bounded by a closed surface S . The eigenfunctions of the Laplace operator for zero boundary conditions on S are solutions of the problem below. In other contexts the equation is known as the Helmholtz equation.

$$\nabla^2 \psi = -\lambda \psi \text{ in } V, \text{ with } \psi|_S = 0.$$

(a) Show that eigenfunctions associated with distinct eigenvalues are orthogonal. That is,

$$\text{if } \nabla^2 \psi_1 = -\lambda_1 \psi_1 \text{ and } \nabla^2 \psi_2 = -\lambda_2 \psi_2, \text{ and if } \lambda_1 \neq \lambda_2, \text{ then } \iiint_V \psi_1 \psi_2 \, d\tau = 0.$$

(b) Derive a Rayleigh Quotient and use it to show that all of the eigenvalues are positive.

(3) (20 points) This problem deals with the function $f(x) = e^{-|x|}(1+x^2)$ and with its Fourier transform.

(a) (7 points) Find the Fourier transform $\bar{f}(k)$ of $f(x)$ by using Mathematica to evaluate the integral defining the transform.

(b) (6 points) Use the definition of the Fourier transform to show that in general

$$\int_{-\infty}^{\infty} f(x) \, dx = \bar{f}(0). \text{ Verify this result for the particular } f(x) \text{ of this problem.}$$

(c) (7 points) Verify Parseval's theorem for this transform pair by using Mathematica to evaluate both of the relevant integrals. (Reminder: Parseval's Theorem says that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\bar{f}(k)|^2 \, dk.$$

(4) (20 points) In this problem, you will apply the Fourier Transform to the solution of a simple ordinary differential equation for a function $y(x)$. The equation is

$$y'' - y = -e^{-|x|}(1+x^2), \text{ with } y \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

(a) (8 points) Take the Fourier transform of the equation and find $\bar{y}(k)$, the Fourier transform of $y(x)$.

(b) (8 points) Invert the transform to find $y(x)$. Do this by using Mathematica to evaluate the inversion integral.

(c) (4 points) Plot your solution from $x = -10$ to $x = 10$.

CHALLENGE PROBLEM

This is a variation of the vibrating beam considered in problem (1) above. In this variation you will work with a fourth order Sturm-Liouville system.

A flexible beam occupies the region $0 \leq x \leq L$. The end at $x = 0$ is clamped so that both the displacement and slope remain zero, and the end at $x = L$ is simply supported, which means

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CHALLENGE PROBLEM (continued)

that the displacement is zero, but the beam can rotate freely in response to bending moments from equilibrium. The vertical displacement y is a function of both position x and time t . The equation and boundary conditions governing the motion are

$$\frac{\partial^2 y}{\partial t^2} = -\sigma \frac{\partial^4 y}{\partial x^4}, 0 < x < L \text{ and } t > 0,$$

$$\text{with } y(0,t) = 0, \frac{\partial y}{\partial x}(0,t) = 0, y(L,t) = 0, \text{ and } \frac{\partial^2 y}{\partial x^2}(L,t) = 0.$$

Here $\sigma = (EI)/(\rho A)$, where E is the Young's modulus (relating stress and strain), ρ is the material density, A is the cross-sectional area of the beam, and I is a geometric property called the moment of inertia.

(a) (15 points) As in problem (1) look for normal modes of vibration of the form $y(x,t) = \cos(\omega t)F(x)$, and show that F satisfies the fourth-order Sturm-Liouville system given below, where $\lambda = \omega^2 / \sigma$.

$$\frac{d^4 F}{dx^4} = \lambda F \text{ for } 0 < x < L,$$

$$\text{with } F(0) = 0, F'(0) = 0, F(L) = 0, \text{ and } F''(L) = 0.$$

(b) (15 points) Show that eigenfunctions associated with distinct eigenvalues are orthogonal.

(c) (15 points) Derive a Rayleigh Quotient and show that all of the eigenvalues are positive.

(d) (15 points) Find the general solution of the differential equation. (First hint: Your results will be much less cluttered if you first let $\lambda = \beta^4$. Second hint: $D^4 - \beta^4 = (D^2 - \beta^2)(D^2 + \beta^2)$,

where $D = \frac{d}{dx}$.)

(e) (15 points) By imposing the four boundary conditions on the general solution, show that the eigenvalue equation is $\tanh(z) = \tan(z)$, where $z = \beta L$.

(f) (15 points) Find the first three values of z and the first three eigenvalues.

(g) (15 points) Using the parameter values given in problem (1), along with the density you found there, determine the frequency in Hz of the fundamental mode in the present case.