

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #9 2011

Assignments handed in by 6 PM on Wednesday Nov. 16 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday Nov. 17 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday Nov. 17.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
VI. UNBOUNDED DOMAINS		
6.3 Laplace Equation	W,Th Nov 2,3	10.6.3
6.4 Similarity Methods	F Nov 4	---
6.5 Kelvin's Estimate of the Earth's Age	M Nov 7	---
Exam Review	W Nov 9	---
Exam #2	Th Nov 10	
VII. PROBLEMS IN SPHERICAL COORDINATES		
7.1 Review of Power Series	F Nov 11	---
7.2 Power Series Solutions of Ordinary Differential Equations	M Nov 14	---

PROBLEMS

(1) (15 points)

(a) (5 points) Consider the function $f(x) = e^{-(x/b)^2}$ where b is a positive constant. Find Δ_x , the e-folding half width of this pulse – i.e., the value of x at which the amplitude of f is $(1/e)$ times its maximum amplitude.

(b) (5 points) Use the table of transforms handed out in class to find $\tilde{f}(k)$, the Fourier transform of f . Find Δ_k , the e-folding half width of $\tilde{f}(k)$ – i.e., the value of k at which the amplitude of \tilde{f} is $(1/e)$ times its maximum amplitude.

(c) (5 points) Show that $\Delta_x \Delta_k = 2$ for any value of the parameter a . Use this to discuss the relation between the pulse width in x -space and the pulse width in k -space.

Laplace Equation in an Infinite Strip

(2) (35 points) Consider the boundary value problem given below for the steady-state temperature $T(x,y)$ in a thermally conducting solid in the shape of an infinite strip of width b .

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < b,$$

$$\text{with } \frac{\partial T}{\partial y}(x,0) = 0, \text{ and } T(x,b) = \frac{T_0 a^2}{a^2 + x^2}.$$

Here T_0 and a are positive constants.

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(a) (5 points) Put the problem in dimensionless form by introducing the following dimensionless variables: $\hat{x} = x/a$, $\hat{y} = y/a$, and $\hat{T} = T/T_0$. In the remainder of the problem you may drop the hats on the scaled variables. The only parameter appearing in your dimensionless formulate will be the location of the upper boundary, b/a . Call this parameter σ .

(b) (10 points) Use the Fourier transform in x to obtain an explicit solution for $\hat{T}(k, y)$, the Fourier transform of $T(x, y)$ with respect to x . Write down the inversion integral which gives $T(x, y)$ in terms of $\hat{T}(k, y)$.

(c) (10 points) Using the inversion integral of part (b), show that the temperature T on the boundary $y = 0$ can be put in the form

$$T(x, 0) = \int_0^{\infty} \frac{e^{-k} \cos(kx)}{\cosh(k\sigma)} dk.$$

(d) (10 points) For $\sigma = 1/2$, plot $T(x, 0)$ for $-6 \leq x \leq 6$. To get the values of T , use NIntegrate to evaluate the integral expression found in part (c).

Initial Value Problem for the Wave Equation

(3) (25 points) In our previous work with the wave equation, we were concerned with bounded regions and standing waves. In the present problem we look at propagating waves in one space dimension. You can think of these as waves on a string of infinite length. We will solve the problem when the string is initially displaced, but when there is no initial velocity. You may remember from elementary physics what happens in that case: the initial displacement splits into two pieces, each with half the amplitude of the initial displacement, and with one piece moving to the left and one piece moving to the right. Both move with speed c , where the wave speed c is a parameter appearing in the equation. The most surprising part of that solution is that the shape of each moving piece is unchanged from the shape of the initial disturbance. We will make the Fourier Transform tell us all of this.

Here is the initial value problem to be solved.

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad \Phi(x, 0) = f(x) \quad \text{and} \quad \frac{\partial \Phi(x, 0)}{\partial t} = 0.$$

(a) (10 points) Take the Fourier transform of the equation, solve the resulting second order constant coefficient ordinary differential equation, and impose the initial condition to get

$$\tilde{\Phi}(k, t) = \tilde{g}(k, t) \tilde{f}(k), \quad \text{where} \quad \tilde{g}(k, t) = \cos(kct).$$

(b) (5 points) By the convolution theorem, show that your solution may be expressed as

$$\Phi(x, t) = \int_{-\infty}^{\infty} g(x - x', t) f(x') dx', \quad \text{where} \quad g \text{ is the Fourier inverse of } \tilde{g}.$$

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(3) (continued)

(c) (5 points) By applying the inversion integral to \tilde{g} , show that

$$g(x,t) = \frac{1}{2}[\delta(x+ct) + \delta(x-ct)], \text{ where } \delta \text{ is the Dirac delta function.}$$

(d) (5 points) Combine your results of parts (b) and (c) to derive the form of the solution that shows that the original wave splits into two parts, one propagating to the left and one to the right, both without change of shape.

Power Series Solution of Ordinary Differential Equations

(4) (25 points) Solve each of the differential equations given below for $y(x)$ with a power series expansion. Unless otherwise stated, the expansion is to be taken about $x = 0$. In each case, find the first four nonzero terms in the series.

(a) (8 points) $y'' + 4xy' + 4y = 0$, $y(0) = 1$, $y'(0) = -1$.

(b) (8 points) $y'' - 16y = 0$, $y(0) = 1$, $y'(0) = 5$.

(c) (9 points) $xy'' + 3y' + y = 0$, $y(1) = 1$, $y'(1) = 0$, expansion about $x = 1$.

CHALLENGE PROBLEM

In this problem you will solve the heat equation, with a given initial condition, in one space dimension in an infinite slab. You will use the Fourier transform and convolution to get a solution for a general initial condition. Here is the problem.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad -\infty < x < \infty, t > 0, \text{ with } T(x,0) = f(x).$$

You may assume that the temperature T and the initial temperature $f(x)$ have Fourier transforms.

(a) (20 points) As a first step in the solution, take the Fourier transform of both the equation and the initial condition. You should get a first-order ordinary differential equation in time for the Fourier transform $\tilde{T}(k,t)$. Solve this equation and impose the initial condition to get the solution for $\tilde{T}(k,t)$.

(b) (20 points) The Fourier transform \tilde{T} has the form $\tilde{T}(k,t) = \tilde{g}(k,t)\tilde{f}(k)$ where you know \tilde{g} explicitly from part (a). Find $g(x,t)$, the inverse transform of \tilde{g} , and then use convolution to write the solution as

$$T(x,t) = \int_{-\infty}^{\infty} g(x-x',t)f(x')dx'.$$

(c) (20 points) Use your solution to prove that if $m < f(x) < M$ for all x , then $m < T(x,t) < M$ for all x and t . (In the absence of sources this makes sense physically, but you are being asked to demonstrate it mathematically from your solution.)