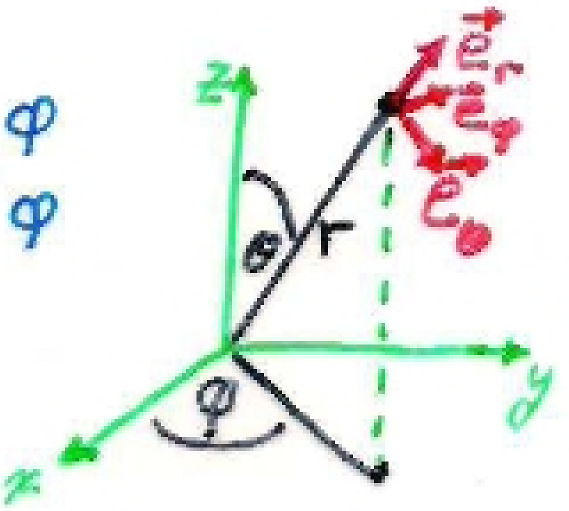
Spherical Polar Coordinates

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$



$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$$
$$\varphi = \tan^{-1}(y/x)$$

$$\vec{i}\cdot\vec{\nabla} = \frac{\partial}{\partial x}$$
$$\vec{j}\cdot\vec{\nabla} = \frac{\partial}{\partial y}$$
$$\vec{k}\cdot\vec{\nabla} = \frac{\partial}{\partial z}$$

$$\vec{e}_r\cdot\vec{\nabla} \propto \frac{\partial}{\partial r}$$
$$\vec{e}_\theta\cdot\vec{\nabla} \propto \frac{\partial}{\partial \theta}$$
$$\vec{e}_\varphi\cdot\vec{\nabla} \propto \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} =$$
$$\sin\theta\cos\varphi\;\vec{i}\cdot\vec{\nabla} + \sin\theta\sin\varphi\;\vec{j}\cdot\vec{\nabla} + \cos\theta\;\vec{k}\cdot\vec{\nabla}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} =$$
$$r\cos\theta\cos\varphi\;\vec{i}\cdot\vec{\nabla} + r\cos\theta\sin\varphi\;\vec{j}\cdot\vec{\nabla} - r\sin\theta\;\vec{k}\cdot\vec{\nabla}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z} =$$
$$-r\sin\theta\sin\varphi\;\vec{i}\cdot\vec{\nabla} + r\sin\theta\cos\varphi\;\vec{j}\cdot\vec{\nabla}$$

Normalized (unit vectors)

$$\vec{e}_r = \sin\theta\cos\varphi\;\vec{i} + \sin\theta\sin\varphi\;\vec{j} + \cos\theta\;\vec{k}$$
$$\vec{e}_\theta = \cos\theta\cos\varphi\;\vec{i} + \cos\theta\sin\varphi\;\vec{j} - \sin\theta\;\vec{k}$$
$$\vec{e}_\varphi = -\sin\varphi\;\vec{i} + \cos\varphi\;\vec{j} + 0\;\vec{k}$$

$$\vec{i} = \sin\theta\cos\varphi\;\vec{e}_r + \cos\theta\cos\varphi\;\vec{e}_\theta - \sin\varphi\;\vec{e}_\varphi$$
$$\vec{j} = \sin\theta\sin\varphi\;\vec{e}_r + \cos\theta\sin\varphi\;\vec{e}_\theta + \cos\varphi\;\vec{e}_\varphi$$
$$\vec{k} = \cos\theta\;\vec{e}_r - \sin\theta\;\vec{e}_\theta + 0\;\vec{e}_\varphi$$

(transposed rotation matrix)

Basis transformation $\vec{e}_i \longrightarrow \vec{e}_i{}'$

Vector coordinates $A_i \equiv \vec{A}\cdot\vec{e}_i \longrightarrow A_i' \equiv \vec{A}\cdot\vec{e}_i{}'$

Direction cosines: $\frac{A_i}{A} = \frac{\vec{A}\cdot\vec{e}_i}{A} = \cos(\vec{A},\vec{e}_i)$

$$A_j' = \vec{e}_j{}'\cdot\vec{A} = \left[\cos(j',1)\vec{e}_1 + \cos(j',2)\vec{e}_2 + \cos(j',3)\vec{e}_3\right]\cdot\vec{A}$$

$$= \cos(j',1)\,A_1 + \cos(j',2)\,A_2 + \cos(j',3)\,A_3$$

or $A_j' = \sum_k \lambda_{jk} A_k$, $\lambda_{jk} \equiv \cos(\vec{e}_j{}', \vec{e}_k) \equiv \vec{e}_j{}'\cdot\vec{e}_k$

or
$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$\underline{A}' = \underline{\underline{\lambda}}\,\underline{A}$$ (matrix notation)

Similarly, $A_i = \vec{e}_i\cdot\vec{A} = \left[\cos(1',i)\vec{e}_1{}' + \cos(2',i)\vec{e}_2{}' + \cos(3',i)\vec{e}_3{}'\right]\cdot\vec{A}$

$$= \lambda_{1i}A_1' + \lambda_{2i}A_2' + \lambda_{3i}A_3'$$

$$= \sum_j \lambda_{ji} A_j'$$

or $\underline{A} = \underline{\underline{\lambda}}^t\,\underline{A}'$ (transposed matrix)

Note $\underline{A} = \underline{\underline{\lambda}}^t(\underline{\underline{\lambda}}\,\underline{A}) = (\underline{\underline{\lambda}}^t\underline{\underline{\lambda}})\,\underline{A}$ for all vectors $\underline{A}$

$\lambda^t\lambda = I \iff \lambda^t = \lambda^{-1}$ ("orthogonal" matrix)

$(\mathrm{Det}\,\lambda)^2 = 1$ $\quad$ $\mathrm{Det}\,\lambda = \pm 1$

$+1 \leftrightarrow$ simple rotations $\Leftrightarrow$ "special orthogonal" matrices

$-1 \leftrightarrow$ inversion also

$\lambda_{ij} = \cos(\vec{e}_i{}', e_j)$

[In Euclidean space, you can compare, add, or subtract vectors at different points.]

Derivative of vector field $\vec{A}$ along a curve

$$\frac{d\vec{A}}{ds} \equiv \lim_{\Delta s \to 0} \frac{\vec{A}(s+\Delta s) - \vec{A}(s)}{\Delta s}$$

Motion of a particle $\vec{r}(t)$

$$\vec{v} \equiv \frac{d}{dt}\vec{r} \quad \text{and} \quad \vec{a} \equiv \frac{d\vec{v}}{dt}$$

In Cartesian coordinates: $\vec{i}, \vec{j}, \vec{k}$ are constant.

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$
$$\vec{v} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j} + \dot{z}\,\vec{k}$$
$$\vec{a} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k}$$

$\left[\dot{x} \equiv \frac{dx}{dt}, \text{ etc.}\right]$

In polar coordinates

$$\dot{\vec{e}}_r = \dot{\theta}\,\vec{e}_\theta + \sin\theta\,\dot{\varphi}\,\vec{e}_\varphi$$
$$\dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_r + \cos\theta\,\dot{\varphi}\,\vec{e}_\varphi$$
$$\dot{\vec{e}}_\varphi = -\sin\theta\,\dot{\varphi}\,\vec{e}_r - \cos\theta\,\dot{\varphi}\,\vec{e}_\theta$$

[See details on next slide]

$$\vec{r} = r\,\vec{e}_r$$
$$\vec{v} = \dot{r}\,\vec{e}_r + r\,\dot{\vec{e}}_r = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\sin\theta\,\dot{\varphi}\,\vec{e}_\varphi$$
$$\vec{a} = \frac{d}{dt}\vec{v} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2)\,\vec{e}_r$$
$$+ (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2)\,\vec{e}_\theta$$
$$+ (r\sin\theta\,\ddot{\varphi} + 2\dot{r}\sin\theta\,\dot{\varphi} + 2r\dot{\theta}\cos\theta\,\dot{\varphi})\,\vec{e}_\varphi$$