Notes on Bayesian Games†

ECON 201B - Game Theory

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1 Bayesian games

So far we have been assuming that everything in the game was common knowledge for everybody playing. But in fact players may have private information about their own payoffs, about their type or preferences, etc. The way to modelling this situation of asymmetric or incomplete information is by recurring to an idea generated by Harsanyi (1967). The key is to introduce a move by the Nature, which transforms the uncertainty by converting an incomplete information problem into an imperfect information problem.

The idea is the Nature moves determining players' types, a concept that embodies all the relevant private information about them (such as payoffs, preferences, beliefs about other players, etc)

Definition 1 *A Bayesian Game is a game in normal form with incomplete information that consists of:*

1) Players $i \in \{1, 2, .., I\}$

2) Finite action set for each player $a_i \in A_i$

3) Finite type set for each player $\theta_i \in \Theta_i$

4) A probability distribution over types $p(\theta)$ *(common prior beliefs about the players' types)*

5) Utilities $u_i : A_1 \times A_2 \times ... \times A_I \times \Theta_1 \times \Theta_2 \times ... \Theta_I \rightarrow \mathbb{R}$

It is important to discuss a little bit each part of the definition.

Players' types contain all relevant information about certain player's private characteristics. The type θ_i is only observed by player i, who uses this information both to make decisions and to update his beliefs about the likelihood of opponents' types (using the conditional probability $p(\theta_{-i}|\theta_i)$)

Combining actions and types for each player it's possible to construct the strategies. Strategies will be given by a mapping from the type space to the

[0] † These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

action space, $s_i : \Theta_i \to A_i$, with elements $s_i(\theta_i)$. In words a strategy may assign different actions to different types.

Finally, utilities are calculated by each player by taking expectations over types using his or her own conditional beliefs about opponents' types. Hence, if player i uses the pure strategy s_i, other players use the strategies s_{-i} and player i's type is θ_i, the expected utility can be written as

$$Eu_i(s_i|s_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i)$$

2 Bayesian Nash Equilibrium (BNE)

A Bayesian Nash Equilibrium is basically the same concept than a Nash Equilibrium with the addition that players need to take expectations over opponents' types. Hence

Definition 2 *A Bayesian Nash Equilibrium (BNE) is a Nash Equilibrium of a Bayesian Game, i.e.*

$$Eu_i(s_i|s_{-i}, \theta_i) \geq Eu_i(s_i'|s_{-i}, \theta_i)$$

for all $s_i'(\theta_i) \in S_i$ *and for all types* θ_i *occurring with positive probability*

Theorem 3 *Every finite Bayesian Game has a Bayesian Nash Equilibrium*

3 Computing BNE

3.1 Example 1

Consider the following Bayesian game:

1) Nature decides whether the payoffs are as in Matrix I or Matrix II, with equal probabilities

2) ROW is informed of the choice of Nature, COL is not

3) ROW chooses U or D, COL chooses L or R (choices are made simultaneously)

4) Payoffs are as in the Matrix chosen by Nature

For each of these games, find all the Bayesian Nash equilibria. Write each equilibrium in mixed behavioral strategies.

Matrix I

	L	R
U	1, 1	0, 0
D	0, 0	0, 0

Matrix II

	L	R
U	0, 0	0, 0
D	0, 0	2, 2

3.1.1 Pure strategy BNE

We will first collapse the incomplete information problem as a static extended game with all the possible strategies (call it $\widehat{\Gamma}$). It can be shown, following Harsanyi, that the Nash Equilibrium in $\widehat{\Gamma}$ is the same equilibrium of the imperfect game presented. The idea is to collapse a game such that all the ways the game can follow is considered in the extended game $\widehat{\Gamma}$.

The first step, as always is to determine the strategies for each player.

COL has only two strategies (L and R) because he does not know in which matrix the game is played.

ROW knows in which Matrix the game occurs, and the strategies are UU (play U in case of being in Matrix I and U in case he is in Matrix II), UD, DU and DD.

Knowing the probability (a half) the Nature locate the game in each matrix (which is needed to obtain expected payoffs), the new extended game $\widehat{\Gamma}$ can be written as:

	L	R
UU	$\underline{\frac{1}{2}}, \underline{\frac{1}{2}}$	0, 0
UD	$\underline{\frac{1}{2}}, \frac{1}{2}$	$\underline{1}, \underline{1}$
DU	0, 0	0, 0
DD	0, 0	$\underline{1}, \underline{1}$

Recall DU is a dominated strategy for ROW. After eliminating that possibility, the game has three pure Nash Equilibrium $\{(UU, L); (UD, R); (DD, R)\}$ (shown by squares in the Matrix above)

3.1.2 Mixed strategy BNE

In order to obtain the mixed strategies we will make another kind of analysis and try to replicate the three pure BNE obtained before.

Assume the probabilities of playing each action are as shown in the matrices below (y is the probability COL plays L (not conditional on the matrix since this is an information COL does not have), x is the probability ROW plays U if the game is in Matrix I and z is the probability ROW plays U if the game is in Matrix II).

Matrix I

		y	$(1-y)$
		L	R
x	U	1, 1	0, 0
$(1-x)$	D	0, 0	0, 0

Matrix II

		y	$(1-y)$
		L	R
z	U	0, 0	0, 0
$(1-z)$	D	0, 0	2, 2