Chapter 3

Dynamic Behavior

In this chapter we give a broad discussion of the behavior of dynamical systems. We focus on systems modeled by differential equations, but consider the general nonlinear case. This allows us to discuss equilibrium points, stability, limit cycles and other key concepts of dynamical systems. We also introduce some measures of performance for input/output systems that provide additional tools for characterizing dynamic behavior.

3.1 Solving Differential Equations

In the last chapter, we saw that one of the methods of modeling dynamical systems is through the use of ordinary differential equations (ODEs). A state space, input/output system has the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x), \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^q$ is the output. The smooth maps $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^q$ represent the dynamics and measurements for the system. We will focus in this text on single input, single output (SISO) systems, for which $p = q = 1$.

We begin by investigating systems in which the input has been set to a function of the state, $u = \alpha(x)$. This is one of the simplest types of feedback, in which the system regulates its own behavior. The differential equations in this case become

$$\dot{x} = f(x, \alpha(x)) = F(x). \tag{3.2}$$

As before, we write $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ for the state vector.

In order to understand the dynamic behavior of this system, we need to analyze the features of the solutions of this equation. While in some simple situations we can write down the solutions in analytical form, more often we must rely on computational approaches. We begin by describing the class of solutions for this problem.

Initial Value Problems

We say that $x(t)$ is a *solution* of the differential equation (3.2) on the time interval $t_0 \in \mathbb{R}$ to $t_f \in \mathbb{R}$ if

$$\dot{x}(t) = F(x(t)) \quad \text{for all } t_0 \leq t < t_f.$$

A given differential equation may have many solutions. We will most often be interested in the *initial value problem*, where $x(t)$ is prescribed at a given time $t_0 \in \mathbb{R}$ and we wish to find a solution valid for all *future* time, $t > t_0$.

We say that $x(t)$ is a solution of the differential equation (3.2) with initial value $x_0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}$ if

$$x(t_0) = x_0 \quad \text{and} \quad \dot{x}(t) = F(x(t)) \quad \text{for all } t_0 \leq t < t_f.$$

For most differential equations we will encounter, there is a *unique* solution that is defined for $t_0 \leq t \leq t_f$. The solution may defined for all time $t > t_0$, in which case we take $t_f = \infty$. Because we will primarily be interested in solutions of the initial value problem for ODEs, we will often refer to this simply as the solution of an ODE.

We will usually assume that t_0 is equal to 0. In the case when F is independent of time (as in equation (3.2)), we can do so without loss of generality by choosing a new independent (time) variable, $\tau = t - t_0$.

Example 3.1 *(Damped oscillator).* Consider a damped, linear oscillator, introduced in Example 2.3. The equations of motion for the system are

$$m\ddot{q} + b\dot{q} + kq = 0,$$

where q is the displacement of the oscillator from its rest position. We assume that $b^2 < 4km$, corresponding to a lightly damped system (the reason for this particular choice will become clear later). We can rewrite this in state space form by setting $x_1 = q$ and $x_2 = \dot{q}$, giving

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2.\end{aligned}$$

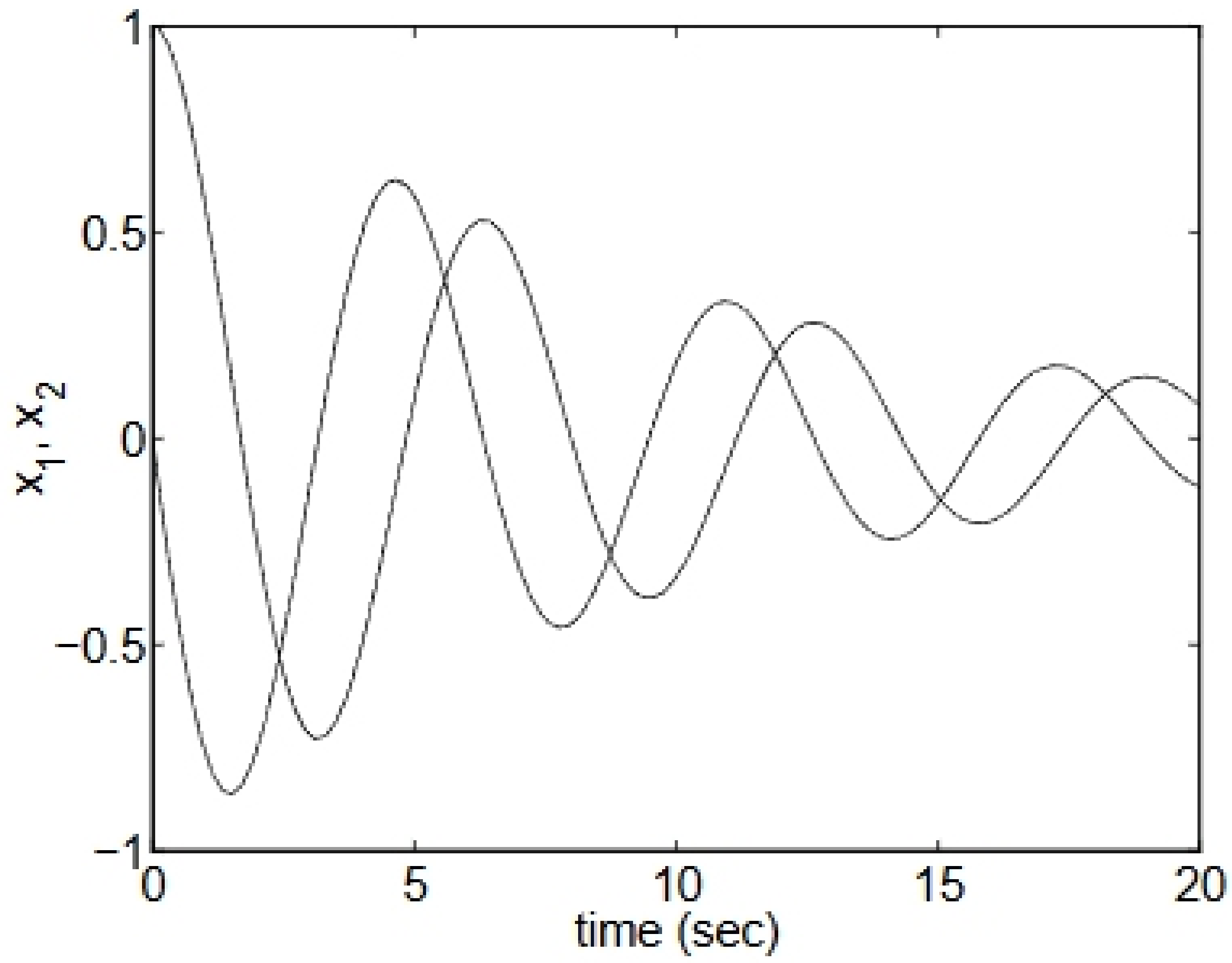

Figure 3.1: Response of the damped oscillator to the initial condition $x_0 = (1, 0)$.

In vector form, the right hand side can be written as

$$F(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{bmatrix}.$$

The solution to the initial value problem can be written in a number of different ways and will be explored in more detail in Chapter 4. Here we simply assert that the solution can be written as

$$\begin{aligned} x_1(t) &= e^{-\frac{bt}{2m}} \left(x_{10} \cos \omega t + (x_{10} + \frac{x_{20}}{\omega}) \sin \omega t \right) \\ x_2(t) &= e^{-\frac{bt}{2m}} \left(\left(x_{20} + x_{10}(\omega - \frac{b}{2}) \right) \cos \omega t + \left(x_{10}\omega + \frac{b}{2}(x_{10} + \frac{x_{20}}{\omega}) \right) \sin \omega t \right) \end{aligned}$$

where $x_0 = (x_{10}, x_{20})$ is the initial condition and $\omega = \sqrt{4km - b^2}/2m$. This solution can be verified by substituting it into the differential equation. We see that the solution is explicitly dependent on the initial condition and it can be shown that this solution is unique. A plot of the initial condition response is shown in Figure 3.1. We note that this form of the solution only holds for $b^2 - 4km < 0$, corresponding to an "underdamped" oscillator.