MATH 2300 - Calculus III

October 24, 2008

Exam 3 Solutions

Name: ______________________________

No Calculators

1. (10 points) Find the volume of the solid above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.

   Solve this in cylindrical coordinates: z is bounded below by $z = r^2$ and above by $z = r$. We solve for the intersection of these two curves $r^2 = r$ and find that $0 \leq r \leq 1$. Finally we integrate:

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{r} r dz dr d\theta = \pi/6$$

2. (15 points) Let S be the triangular region with vertices (0,0), (1,1), and (0,1) and $T = T(u, v) = \langle x(u, v), y(u, v)\rangle = \langle u^2, v\rangle$

   (a) (7 points) Sketch the image of the set S under the transformation T

   Remember that this is a mapping from the u, v-plane to x, y-plane. With this in mind, we use $x = u^2$, and $y = v$ for each of the three lines of the triangle: first the $x = 0$ line, which goes from $0 \leq y \leq 1$. For this line we have $0 = x = u^2$ and thus $u = 0$ and $0 \leq v \leq 1$. For the $y = 1$ line, we have $v = 1$ and $0 \leq u \leq 1$ (e.g. a straight line at $y = 1$). Finally, we need to see what happens to $v = u$. This is simply $\sqrt{x} = y$.

   (b) (8 points) Find the Jacobian of the transformation

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u$$

3. (15 points) Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x, y, z) dz dy dx$$

   as an iterated integral in the order $dx\, dy\, dz$.

   For me the easiest way to do this is by drawing pictures. We see that in the y, z plane, we have the region above $z = 0$ and $z = 1 - y$. In the x, y plane, we're bounded below by the parabola $y = x^2$ and above by $y = 1$. To integrate in x first we go from $x = -\sqrt{y}$ to $x = \sqrt{y}$. Then we integrate in y, which is bounded below by $y = 0$ and above by $y = 1 - z$. Finally we integrate on $0 \leq z \leq 1$. So we have

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx dy dz$$

4. (10 points) Find the gradient vector field of $f(x, y) = xe^{xy}$.

   We simply take the gradient of $f(x, y)$: $\nabla f = \mathbf{F} = \langle e^{xy} + xye^{xy}, xe^{xy}\rangle$

5. (15 points) Show that $\mathbf{F}(x, y, z) = \langle e^y, xe^y + e^z, ye^z\rangle$ is conservative and use this fact to find the work done to move a particle along the the curve C, the line segment from (0,2,0) to (4,0,3).

First, we check by taking the curl of $\mathbf{F}$:

$$\text{curl } F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y + e^z & ye^z \end{vmatrix} = (e^z - e^z)i - (0-0)j + (e^y - e^y)k = (0,0,0)$$

Thus $\mathbf{F}$ is conservative. Next we solve for f by noting $\nabla f = \mathbf{F}$ means that $\frac{\partial}{\partial x} f = e^y$. Integrating we get $f(x,y,z) = xe^y + g(y,z)$. Next, taking the partial in y we get $xe^y + \frac{\partial}{\partial y} g(y,z)$ but we're given that this is equal to $xe^y + e^z$ so $\frac{\partial}{\partial y} g(y,z) = e^z$.. Integrating this with respect to y we get $g(y,z) = ye^z + C(z)$. Finally, we have $f(x,y,z) = xe^y + ye^z + g(z)$. but differentiating this with respect to z gives $f_z = ye^z + g'(z)$. Since we know that $f_z = ye^z$, $g'(z) = 0$, and $f(x,y,z) = xe^y + ye^z$.

Finally, to evaluate the integral, we simply evaluate $f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

6. (10 points) Let D be a region for which Green's theorem holds (*hint hint*). Suppose f is harmonic; that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on D bounded by the curve Γ. Prove that

$$\int_\Gamma \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = 0$$

Here set $P(x,y) = \frac{\partial f}{\partial y}$ and $Q(x,y) = -\frac{\partial f}{\partial x}$. By Green's theorem:

$$\int_\Gamma \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int\int -\frac{\partial}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial y}$$

but the integrand is $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ so the integral is zero.

7. (15 points) Evaluate the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the path C, traversed counter -clockwise around the triangle with vertices (1,0),(0,1),(-1,0) and where $\mathbf{F}(x,y) = \langle 3 + 2y, 4 - 5x \rangle$.

This is the bonus problem simplified. Simply apply Green's theorem to the function $\mathbf{F}$ to get $\int\int_D (-5 - 2) dA = -7 \int_D dA$. Here the domain has unit area, so the integral is -7.

8. (10 points) If f is a function of three variables that has continuous second-order partial derivatives (*Hint: this means Clairout's Theorem applies*), then show that

$$\text{curl}(\nabla f) = (0,0,0)$$

Here we have

$$\text{curl}\nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} f & \frac{\partial}{\partial y} f & \frac{\partial}{\partial z} f \end{vmatrix},$$

which gives the difference of mixed partial derivatives in each component. By Clairout's theorem, the mixed partial derivatives are equal so the curl is zero.

Alternatively, you could state that all gradient vector fields are conservative, and the curl of a conservative vector field is zero.