[§motivate] 1. Motivating, one-dimensional example

If f and g are increasing functions on $\mathbb{R}$ and μ is a probabilty measure on $\mathcal{B}(\mathbb{R})$ for for $\mu(f^2) < \infty$ and $\mu g^2 < \infty$ then

$$\mu(f)\mu(g) \le \mu(fg)$$

Proof. Expand the left-hand side of

$$\mu^x \mu^y \big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \ge 0$$

□

[§AD] 2. Generalized FKG

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ define

$$x \vee y = (x_1 \vee y_1, \dots, x_n \vee y_n) \qquad \text{and} \qquad x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$$

Write $x \le y$ to mean $x \vee y = x$. Say that a function f on $\mathbb{R}^n$ is increasing if it is an increasing function in each of its arguments (for fixed values of the other arguments). Equivalently, f is increasing if $f(x) \le f(y)$ whenever $x \le y$.

AD <1> **Theorem.** *Suppose $f_1, \dots, f_4$ are nonnegative, Borel-measurable functions on $\mathcal{X}^n$, where $\mathcal{X} \subseteq \mathbb{R}$, for which*

AD-ineq <2>
$$f_1(x) f_2(y) \le f_3(x \vee y) f_4(x \wedge y) \qquad \text{for all } x, y \in \mathcal{X}^n.$$

Let $\mu = \mu_1 \otimes \dots \otimes \mu_n$ be a sigma-finite product measure on $\mathcal{B}(\mathcal{X}^n)$. Then

$$\mu(f_1)\mu(f_2) \le \mu(f_3)\mu(f_4)$$

Proof. Integrate out one coordinate at a time, showing that the key inequality is preserved. Write $x = (X, u)$ and $y = (Y, v)$, where $X = (x_1, \dots, x_{n-1})$ and $Y = (y_1, \dots, y_{n-1})$. Define $\widetilde{f}(X) := \mu_n^u f_i(X, u)$. We need to show that

tf <3>
$$\widetilde{f_1}(X)\widetilde{f_2}(Y) \le \widetilde{f_3}(X \wedge Y)\widetilde{f_4}(X \vee Y)$$

The left-hand side of <3> equals

$$\begin{aligned}
&\mu_n^u \mu_n^v f_1(X, u) f_2(Y, v) \\
&\qquad = \mu_n^u \mu_n^v \big(\{u = v\} f_1(X, u) f_2(Y, v)\big) \\
&\qquad\quad + \mu_n^u \mu_n^v \big(\{u < v\} f_1(X, u) f_2(Y, v) + f_1(X, v) f_2(Y, u)\big)
\end{aligned}$$

The right-hand side of <3> equals

$$\begin{aligned}
&\mu_n^u \mu_n^v f_3(X \wedge Y, u) f_4(X \vee Y, v) \\
&\quad = \mu_n^u \mu_n^v \big(\{u = v\} f_3(X \wedge Y, u) f_4(X \wedge Y, v)\big) \\
&\qquad + \mu_n^u \mu_n^v \big(\{u < v\} f_3(X \wedge Y, u) f_4(X \vee Y, v) + f_3(X \wedge Y, v) f_4(X \vee Y, u)\big)
\end{aligned}$$

On the set $\{u = v\}$, inequality <2> gives

$$f_1(X, u) f_2(Y, v) \le f_3(X \wedge Y, u) f_4(X \vee Y, v)$$

On the set $\{u < v\}$,

$$\begin{aligned}
A &:= f_1(X,u)f_2(Y,v) \le C := f_3(X \wedge Y, u)f_4(X \vee Y, v)\\
B &:= f_1(X,v)f_2(Y,u) \le C\\
AB &= f_1(X,u)f_2(Y,u)f_1(X,v)f_2(Y,v)\\
&\le f_3(X \wedge Y, u)f_4(X \vee Y, u)f_3(X \wedge Y, v)f_4(X \vee Y, v)\\
&= CD \qquad \text{where } D := f_3(X \wedge Y, v)f_4(X \vee Y, u)
\end{aligned}$$

If we can show that

ABCD <4>
$$A + B \le C + D$$

then the inequality <3> will follow by pointwise inequalities on the integrands. Inequality <4> is just a rearrangement of the inequality

$$\begin{aligned}
0 \le (1 - A/C)(1 - B/C) &= 1 - (A+B)/C + (AB)/C^2\\
&\le (C + D - A - B)/C
\end{aligned}$$

□ And so on.

PQ <5> **Corollary.** *Suppose P and Q are probability measures on $\mathcal{B}(\mathcal{X}^n)$ with densities $p = dP/d\mu$ and $q = dQ/d\mu$ with respect to a product measure μ. Suppose*

$$p(x)q(y) \le p(x \wedge y)q(x \vee y) \qquad \text{for all } x, y \in \mathcal{X}^n$$

Then

$$Pf \le Qf$$

for each increasing function f that is both P- and Q-integrable.

Proof. Without loss of generality f is bounded and nonnegative. [Truncate; recenter; Dominated Convergence.] Define

$$\begin{aligned}
f_1(x) &= p(x)f(x)\\
f_2(x) &= q(x)\\
f_3(x) &= p(x)\\
f_4(x) &= q(x)f(x)
\end{aligned}$$

Check that

$$\begin{aligned}
f_1(x)f_2(y) &= f(x)p(x)q(y)\\
&\le f(x \vee y)p(x \wedge y)q(x \vee y) = f_3(x \wedge y)f_4(x \vee y)
\end{aligned}$$

□ Invoke Theorem <1>.

P <6> **Corollary.** *Suppose P is a probability measure with a density $p = dP/d\mu$ with respect to a product measure μ, for which*

$$p(x)p(y) \le p(x \wedge y)p(x \vee y) \qquad \text{for all } x, y \in \mathcal{X}^n$$

If f and g are increasing, P-square integrable functions on $\mathcal{X}^n$ then

$$Pf(x)g(x) \ge (Pf)(Pg)$$

That is, f and g are positively correlated as random variables under P.

Proof. Once again reduce to the case where f is nonnegative. Define

$$\begin{aligned}
f_1(x) &= p(x)f(x)\\
f_2(x) &= p(x)g(x)\\
f_3(x) &= p(x)\\
f_4(x) &= p(x)f(x)g(x)
\end{aligned}$$

Check that

$$f_1(x)f_2(y) = f(x)g(y)p(x)p(y)$$
$$\leq f(x \vee y)g(x \vee y)p(x \wedge y)p(x \vee y) = f_3(x \wedge y)f_4(x \vee y)$$

□ Invoke Theorem <1>.

[§Ising]

3. Application to Ising measures on $\mathbb{Z}^2$

The Ising model gives a joint distribution for an infinite collection of random variables $\{X_i : i \in \mathbb{Z}^2\}$, indexed by the (***sites***) (lattice points) in the lattice $\mathbb{Z}^2$, with each X_i taking values in $\{-1, +1\}$. In fact, the construction of the whole joint distibution is quite subtle. One starts by defining joint (conditional) distributions for $\{X_i : i \in A\}$ for various finite subsets A of $\mathbb{Z}^2$. These distributions satisfy a consistency condition (described in Lemma <14> below) tt enables them to be pasted together to form a joint distribution over all sites in $\mathbb{Z}^2$.

The lattice $\mathbb{Z}^2$ is thought of as the set of vertices in an infinite graph whose edge set $\mathcal{E}$ consists of all all pairs $e = \{i, j\}$ of sites separated by a Eucliden distance 1. For example, the set of neighbors of a site $i = (i_1, i_2)$ is

$$\partial\{i\} := \{(i_1, i_2 + 1), (i_1, i_2 - 1), (i_1 + 1, i_2), (i_1 - 1, i_2)\}.$$

There are four edge with site i as one vertex.

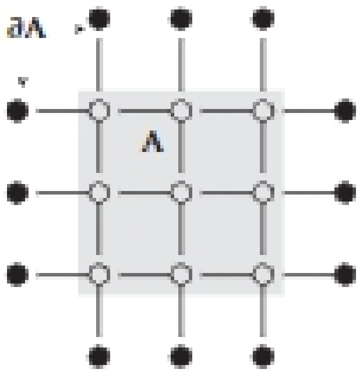
More generally, the boundary ∂A of a set $A \subset \mathbb{Z}^2$ is defined as

$$\partial A := \{j \in A^c : \{i, j\} \in \mathcal{E} \text{ for some } i \text{ in } A\}$$

We could also define ∂A as $\cup\{e : e \in \mathcal{E}_A\}\backslash A$, where $\mathcal{E}_A$ denotes the set of all edges $e = \{i, j\}$ for which at least one vertex (i, or j, or maybe both) is in A.

> REMARK. The terminology seems a little strange to me, because the boundary of a set in the topological sense is not required to be disjoint from A.

neighbors <7> **Example.** Suppose A consists of 9 sites in the form of a 3×3 grid, the vertices represented by the circles inside the shaded region in the following picture.



The boundary ∂A consists of the 12 sites indicated by the circles filled with black. There are 24 edges in $\mathcal{E}_A$: 12 are between pairs of sites in A and 12 are between a site in A and a site in ∂A. □

For each $\beta > 0$ and each $b \in \{-1, +1\}^{\partial A}$ and $x_A \in \{-1, +1\}^A$, define the (conditional) probability that X_A equals x_A by

$$\mathbb{P}_A\{X_A = x_A \mid X_{\partial A} = b\} := p_A(x_A \mid b) := \frac{1}{Z_A(b)} \prod_{\{i,j\} \in \mathcal{E}_A} \exp(\beta x_i x_j)$$