Lecture: 03-10-14 Professor Shabanskaya Calculus 1760-002

Section 7.4.1: Improper Integrals over Unbounded Intervals

Let f be continuous on the intervals given below

1) $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ $(t \neq \infty, -\infty, t \geq a)$

2.) $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ $(t \neq \infty, -\infty, t \leq b)$

3.) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$

Definition: We say an improper intergral is convergent if the limit(s) exist and is (are) finite, otherwise the improper integral is divergent.

Ex: We found that the improper integral $\int_3^{\infty} 2/x^5\,dx$ is convergent.

Consider $\int_a^{\infty} \frac{1}{x^p}\,dx$ for ~~$0<p\leq 1$~~ $p>0$. Let's see for which values of p, $p>0$ such improper intergrals are divergent or convergent. ($f(x) = \frac{1}{x^p}$, $p>0$ is continuous on $[a,\infty)$)

$$\int_a^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^p}\,dx \quad (p>0,\ t\geq a \neq \infty, -\infty)$$

$$\begin{cases} \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_a^t & 0<p<1, \quad \lim_{t\to\infty} \ln|x|\Big|_a^t \quad p=1 \\ \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_a^t & p>1 \end{cases} =$$

$$= \begin{cases} 0<p<1, & \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right) = \lim_{t\to\infty}\frac{t^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1} = \\ p=1, & \lim_{t\to\infty}(\ln|t| - \ln|a|) = \lim_{t\to\infty} \ln|t| - \ln|a| = \infty \\ p>1, & \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1}\right) - \frac{a^{-p+1}}{-p+1} = \lim_{t\to\infty}\frac{1}{t^{p-1}(-p+1)} - \frac{a^{-p+1}}{-p+1} = \end{cases}$$

$= \lim_{t\to\infty} \frac{t^{(-p+1)>0}}{-p+1} - \frac{a^{-p+1}}{-p+1} = \infty$ ($0<p<1$, & $p=1$ are divergent)

$= -\frac{a^{-p+1}}{-p+1} < \infty$

Result: The improper integral $\int_a^{\infty} \frac{1}{x^p}\,dx$, where $f(x)$ is continous on $[a,\infty)$ is convergent when $p>1$ and divergent when $0<p\leq 1$.

Ex: Determine whether the improper integral is convergent or divergent & calculate value if it is convergent.

1.) $\int_2^{\infty} 3x^{-2}\,dx = 3\int_2^{\infty} x^{-2}\,dx = 3\int_2^{\infty}\frac{1}{x^2}\,dx$

a.) $f(x) = 1/x^2$ is continuous on $[2,\infty)$.

b.) Since $p=2>1$, $3\int_2^{\infty}\frac{1}{x^2}\,dx$ is convergent

c.) $\int_2^{\infty} 3x^{-2}\,dx = 3\lim_{t\to\infty}\int_2^t x^{-2}\,dx = 3\lim_{t\to\infty} -\frac{1}{x}\Big|_2^t = 3\lim_{t\to\infty}\left(-\frac{1}{t}\right) + \frac{3}{2} = \boxed{\frac{3}{2}}$

2. $\int_{-\infty}^{\infty} 4x^3 e^{-x^4} dx = \int_{-\infty}^{1} 4x^3 e^{-x^4} dx + \int_{1}^{\infty} 4x^3 e^{-x^4}$

a.) is continuous on $(-\infty, \infty)$

consider $\int_{-\infty}^{1} 4x^3 e^{-x^4} dx = \lim_{t \to -\infty} \int_{-\infty}^{1} 4x^3 e^{-x^4} dx$ $\left(\begin{array}{l} u = -x^4 \\ du = -4x^3 dx \end{array} \right)$

$\lim_{t \to -\infty} -\int e^{u} du$

If $x = t$, $u = -t^4$) If $x = 1$, $u = -1$