Math exam #3 Study Guide

- polar coordinates:
  $dA = r\,dr\,d\theta$ $r = \sqrt{x^2+y^2}$
  $x = r\cos\theta$ r = dist. fr/origin
  $y = r\sin\theta$

cylindrical coordinates:
(θ, r, z) $\iiint$ $r\,dr\,d\theta\,dz$

- Triple integrals
  a) w/ rectangular box:
  $[a,b] \times [c,d] \times [e,f]$
  $$\int_e^f \int_c^d \int_a^b dx\,dy\,dz$$
  b) using z sections:
  $(x,y) \in D$ $u_1(x,y) \leq z \leq u_2(x,y)$ to find region D, set $u_1 = u_2$
  $$\iint_D \int_{u_1}^{u_2} f(x,y,z)\,dA$$

- center of mass $(\bar{x}, \bar{y})$ if ρ is const, assume $\rho = 1$

2D a) total mass $m = \iint_D \rho(x,y)\,dA$

$M_x = \iint_D y \cdot \rho(x,y)\,dA$ $\bar{y} = \frac{M_x}{m}$

$M_y = \iint_D x \cdot \rho(x,y)\,dA$ $\bar{x} = \frac{M_y}{m}$

3D b) total mass $m = \iiint_E \rho(x,y,z)\,dV$

$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$ $\bar{z} = \frac{M_{xy}}{m}$

$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dV$ $\bar{x} = \frac{M_{yz}}{m}$

$M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV$ $\bar{y} = \frac{M_{xz}}{m}$

- spherical system

(ρ, θ, φ)

$z = \rho\cos\varphi$ $\quad x = \rho\sin\varphi\cos\theta$ $\quad \rho \geq 0$

$r = \rho\sin\varphi$ $\quad y = \rho\sin\varphi\sin\theta$ $\quad 0 \leq \theta \leq 2\pi$

$\rho^2 = x^2+y^2+z^2$ $\quad 0 \leq \varphi \leq \pi$

$$\iiint_E f \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

w/ constant ρ ex) $\rho = 1$, get a sphere

w/ $\varphi = \pi/4$, get horiz. circle

w/ $\theta = \pi/4$, get vert. circle

- change of variables

a) 1 variable case:

$$\int_{x=a}^{x=b} f(x)\,dx \underset{②}{\overset{①}{\rightarrow}} \int_{u=c}^{u=d} f(g(u))g'(u)\,du$$

① change limits: $x = a/b \rightarrow u = c/d$

② rewrite f in terms of u $(f(g(u)))$

③ replace dx w/ $g'(u)\,du$

b) 2 variable case:

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}$$

① get u and v equations

② use equations to find y/x

③ plug in constraints to get new constraints

④ set up integral using jacobian and new constraints

c) 3 variable case in 3-D:

$$\iiint_S f(g(u,v,w), h(u,v,w), j(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{array}{c|ccc|} & \partial u & dv & \partial w \\ \partial x & \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y & \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z & \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{array} = \frac{\partial x}{\partial u}\begin{vmatrix} \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial v & \partial z/\partial w \end{vmatrix} - \text{etc.}$$

- Jacobian when cylindrical $(x,y,z) \rightarrow (r,\theta,z)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{array}{c|ccc|} & \partial r & \partial\theta & \partial z \\ \partial x & \cos\theta & -r\sin\theta & 0 \\ \partial y & \sin\theta & r\cos\theta & 0 \\ \partial z & 0 & 0 & 1 \end{array} \rightarrow = r$$

- Line integrals ← length of a curve $= \sqrt{x'(t)^2 + y'(t)^2}$

$\int_a^b F(x(t),y(t)) \underbrace{\|\vec{r}'(t)\| dt}_{ds}$ $a \le t \le b$ for $\int_C F(x,y)\,ds$

Fundamental thm

$$\int_a^b F(x)dx = F(b) - F(a)$$

- to parametrize a circle of radius a:
  - $x = a\cos t$ $0 \le t \le \pi$
  - $y = a\sin t$
- to parametrize line segment from $(x,y) \rightarrow (x_1,y_1)$
  - $x(t) = (1-t)x_0 + tx_1$ $0 \le t \le 1$
  - $y(t) = (1-t)y + ty_1$

- given curve C and linear density $\rho(x,y)$:

mass $m = \int_C \rho\, ds$

$\bar{x} = \frac{1}{m}\int_C x\rho\, ds$ $\bar{y} = \frac{1}{m}\int_C y\rho\, ds$

- other line integrals:

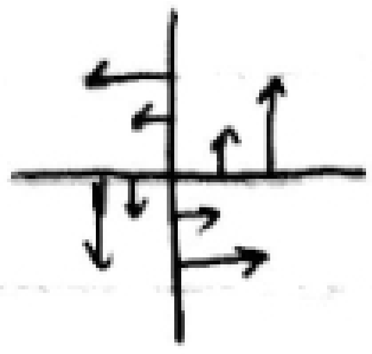
$\int_C f(x,y)dx + g(x,y)dy$

$= \int_a^b f(x(t),y(t))x'(t)dt + g(x(t),y(t))y'(t)dt$

- vector fields

$\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$
$= \langle P(x,y), Q(x,y)\rangle$

map for $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$:



@ $(1,0)$, $\vec{F} = (0,1)$
@ $(0,1)$, $\vec{F} = (-1,0)$

- $\nabla f(x,y) = \langle f_x, f_y \rangle$
- $W = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}'(t)dt = \int_a^b P dx + Q dy$

↑ if $\vec{F} = P\hat{i} + Q\hat{j}$ and $\vec{r} = \langle x(t), y(t)\rangle$