MA123, Chapter 6: Extreme values, Mean Value Theorem, Curve sketching, and Concavity

`Chapter Goals:`

- Apply the Extreme Value Theorem to find the global extrema for continuous function on closed and bounded interval.
- Understand the connection between critical points and local extreme values.
- Understand the relationship between the sign of the derivative and the intervals on which a function is increasing and on which it is decreasing.
- Understand the statement and consequences of the Mean Value Theorem.
- Understand how the derivative can help you sketch the graph of a function.
- Understand how to use the derivative to find the global extreme values (if any) of a continuous function over an unbounded interval.
- Understand the connection between the sign of the second derivative of a function and the concavities of the graph of the function.
- Understand the meaning of inflection points and how to locate them.

`Assignments:`

`Assignment 12` `Assignment 13`
`Assignment 14` `Assignment 15`

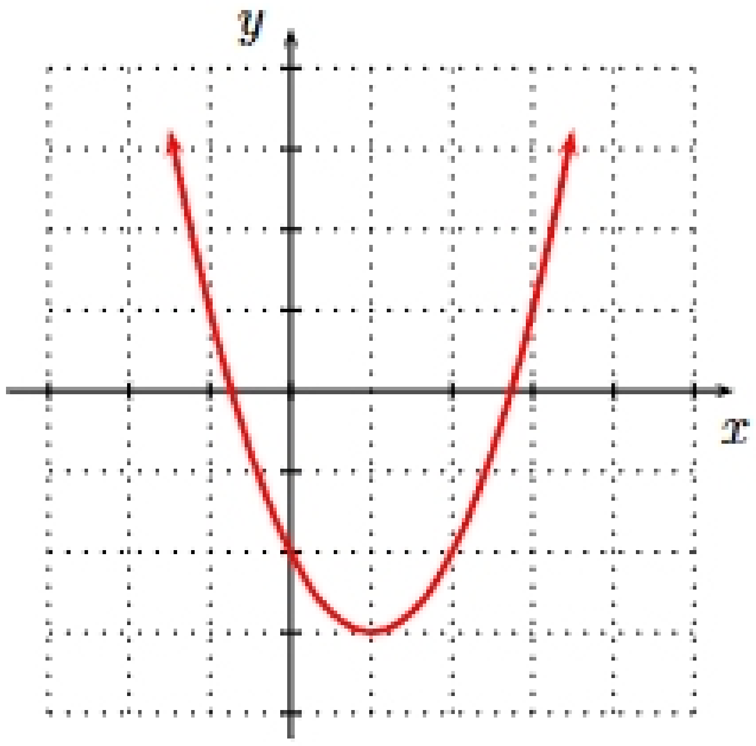
Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task, or figuring out how to perform a task most productively under a given budget and time schedule are some examples of practical real-world applications of Calculus. The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval. This procedure depends on the nature of the interval.

► **Global (or absolute) extreme values:** The largest value a function (possibly) attains on an interval is called its **global (or absolute) maximum value**. The smallest value a function (possibly) attains on an interval is called its **global (or absolute) minimum value**. Both maximum and minimum values (if they exist) are called **global (or absolute) extreme values**.

Example 1(a):
Find the maximum and minimum values for the function

$$f(x) = (x-1)^2 - 3,$$

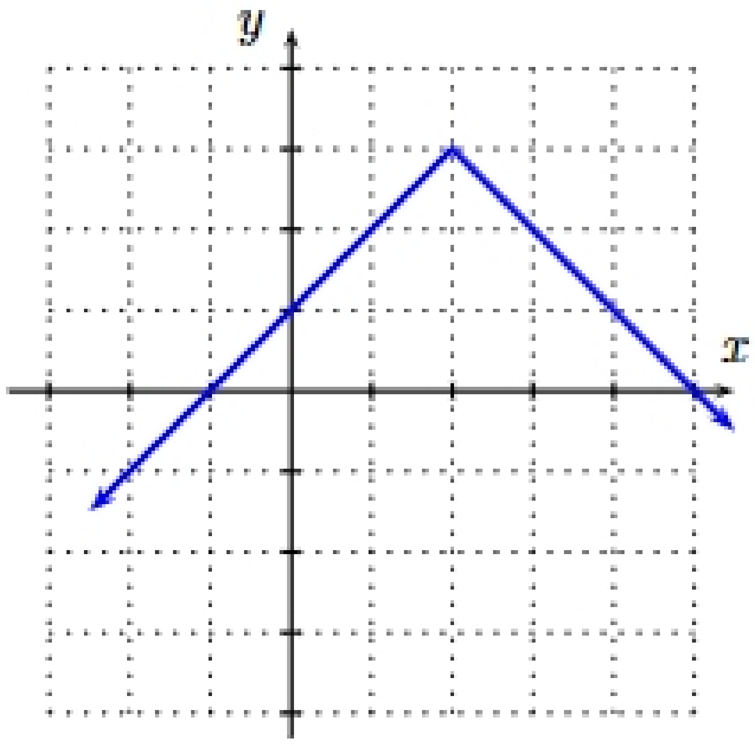
if they exist.



Example 1(b):
Find the maximum and minimum values for the function

$$f(x) = -|x-2| + 3,$$

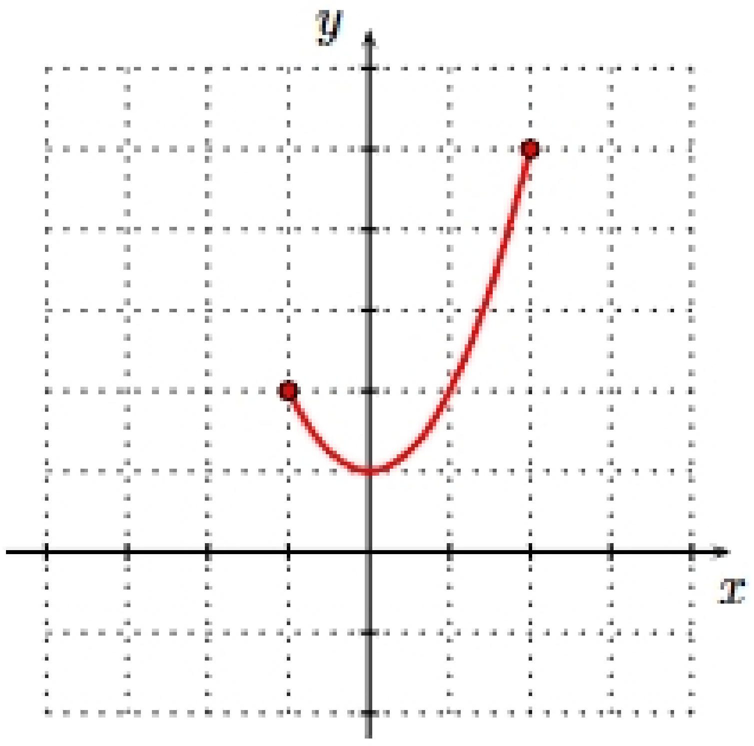
if they exist.



Example 1(c):
Find the maximum and minimum values for the function

$$f(x) = x^2 + 1, \qquad x \in [-1, 2]$$

if they exist.



We first focus on continuous functions on a closed and bounded interval. The question of largest and smallest values of a continuous function f on an interval that is not closed and bounded requires us to pay more attention to the behavior of the graph of f, and specifically to where the graph is rising and where it is falling.

Closed and bounded intervals:

An interval is **closed and bounded** if it has finite length and contains its endpoints.

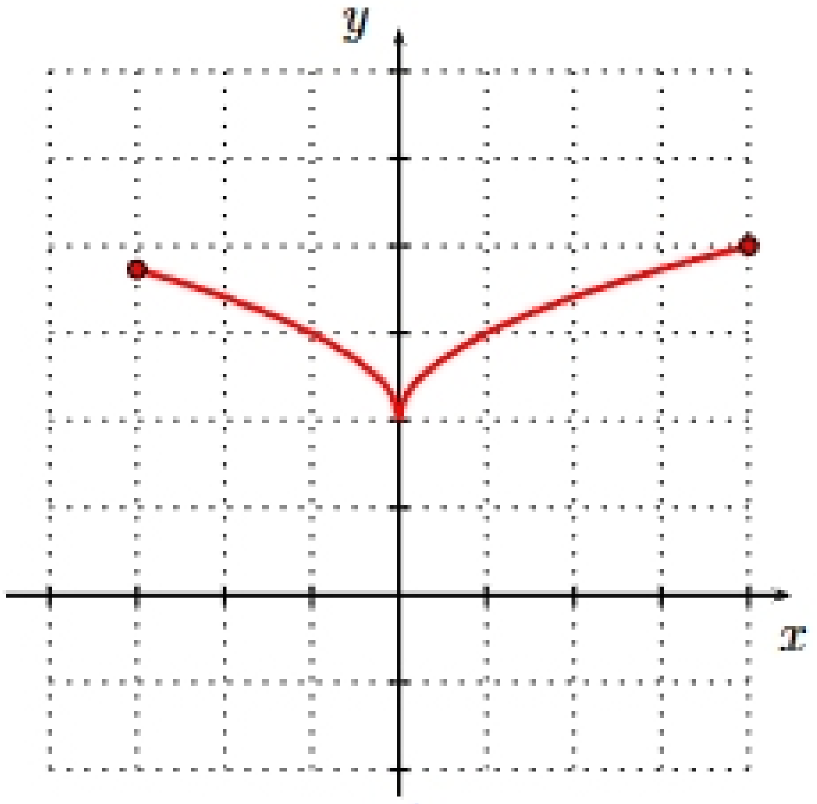
For example, the interval $[-2,5]$ is closed and bounded.

► **The Extreme Value Theorem (EVT):**

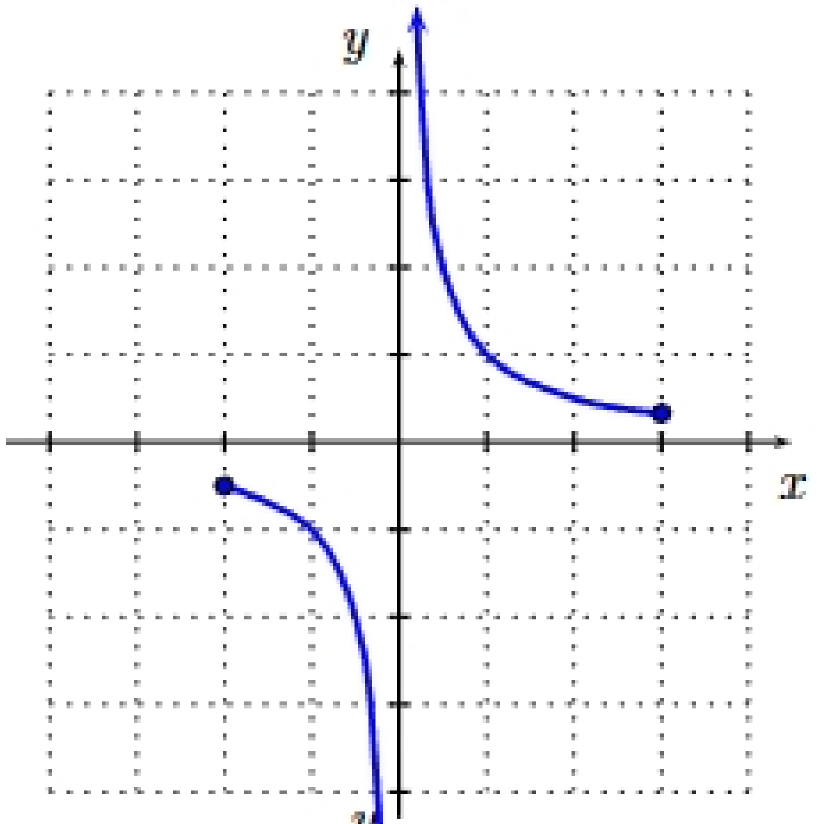
If a function f is continuous on a closed, bounded interval $[a,b]$, then the function f attains a maximum and a minimum value on $[a,b]$.

Example 2(a): Let $f(x) = \begin{cases} 2+\sqrt{x} & \text{if } x > 0 \\ 2+\sqrt{-x} & \text{if } x \leq 0. \end{cases}$

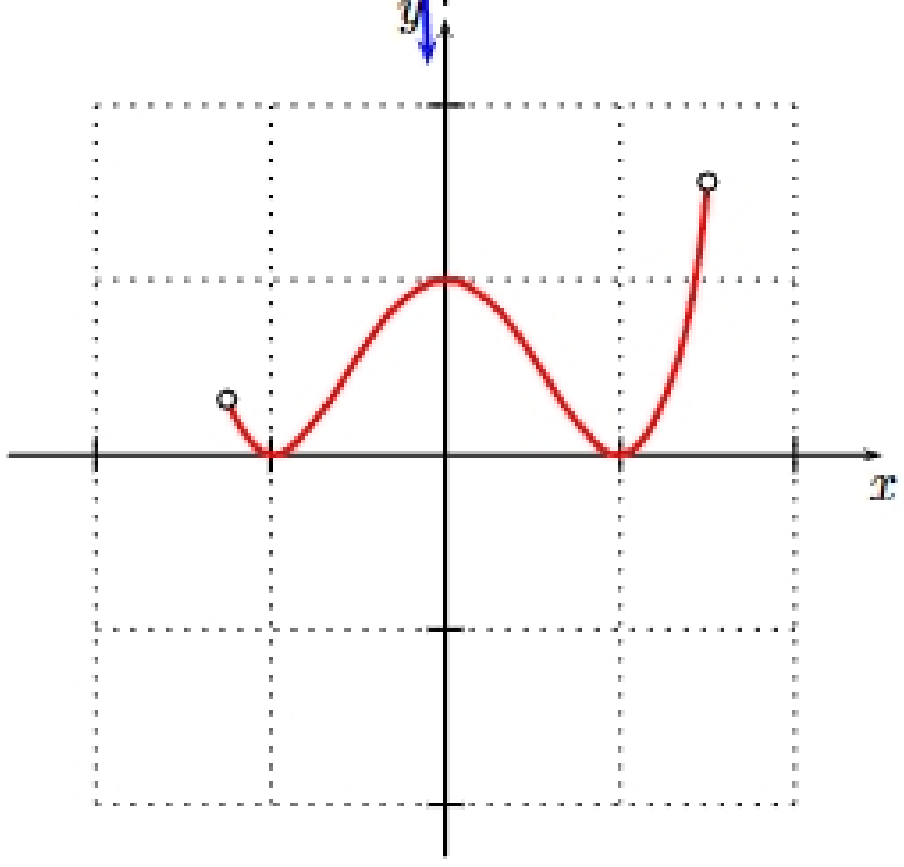
Does $f(x)$ have a maximum and a minimum value on $[-3,4]$? How does this example illustrate the Extreme Value Theorem?



Example 2(b): Let $g(x) = \dfrac{1}{x}$. Does $g(x)$ have a maximum value and a minimum value on $[-2,3]$? Does this example contradict the Extreme Value Theorem? Why or why not?



Example 2(c): Let $h(x) = x^4 - 2x^2 + 1$. Does $h(x)$ have a maximum value and a minimum value on $(-1.25, 1.5)$? Does this example contradict the Extreme Value Theorem? Why or why not?



The EVT is an existence statement; it doesn't tell you how to locate the maximum and minimum values of f.

The following results tell you how to narrow down the list of possible points on the given interval where the function f *might* have an extreme value to (usually) just a few possibilities. You can then evaluate f at these few possibilities, and pick out the smallest and largest value.

▶ **Fermat's Theorem:** Let $f(x)$ be a continuous function on the interval $[a,b]$. If f has an extreme value at a point c strictly between a and b, and if f is differentiable at $x=c$, then $f'(c)=0$.

▶ **Corollary:** Let $f(x)$ be a continuous function on the closed, bounded interval $[a,b]$. If f has an extreme value at $x=c$ in the interval, then either

- $c=a$ or $c=b$;
- $a<c<b$ and $f'(c)=0$;
- $a<c<b$ and f is not differentiable at $x=c$, so that f' is not defined at $x=c$.

Example 3: Find the maximum and minimum values of $f(x)=x^3-3x^2-9x+5$ on the interval $[0,4]$. For which values x are the maximum and minimum values attained?

Example 4: Find the maximum and minimum values of $F(s)=\dfrac{2s+1}{s-6}$ on the interval $[-1,5]$. For which values s are the maximum and minimum values attained?

Example 5: Find the maximum and minimum values of $f(x)=x^{2/3}$ on the interval $[-1,8]$. For which values s are the maximum and minimum values attained?