MA123, Chapter 4: Computing Some Derivatives (pp. 69-82, Gootman)

`Chapter Goals:`
- Understand the derivative as the slope of the tangent line at a point.
- Investigate further the notions of continuity and differentiability.
- Use the definition to calculate some derivatives.
- Use the definition to approximate some derivatives.

`Assignments:` `Assignment 06` `Assignment 07`

In this chapter we explore further the relation between the derivative and the equation of the tangent line at a point. Then we learn how to compute the derivative of some functions using the definition of the derivative. One reason for doing this is to convince you that the rules and formulas for derivatives are not magical. They have a solid foundation and can be explained with just a little bit of effort. Learning should not just be a matter of memorizing mysterious formulas but it should rather be a matter of understanding them.

We start by recalling the following facts that we encountered in Chapter 2:

▶ **Basic facts about derivatives:** The instantaneous rate of change of a function f with respect to x at a general point x is called the *derivative of f at x* and is denoted with $f'(x)$:

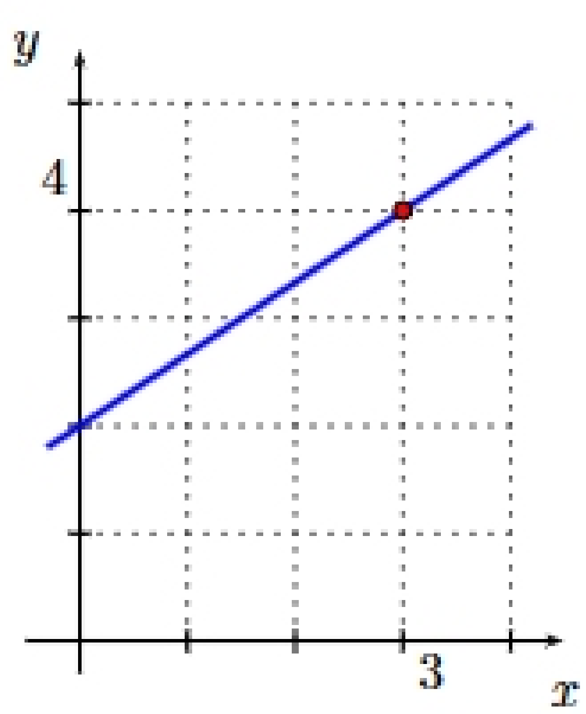
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For a given value x_0, *the derivative of f at x_0, namely $f'(x_0)$, gives the slope of the tangent line to the graph of f at the point $(x_0, f(x_0))$*. Thus, the equation of the tangent line to such a point is given by the formula

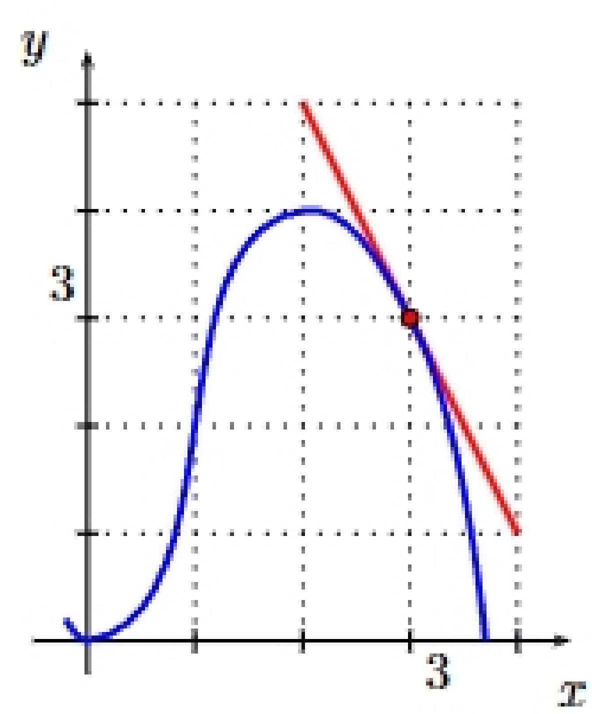
$$y = f(x_0) + f'(x_0)(x - x_0).$$

▶ **Tangent lines, continuity and differentiability:** In the following problems we practice computing equations of tangent lines. Also, we investigate further the notions of continuity and differentiability of a function at a point. Please refer back to Chapter 3 for the corresponding definitions.

Example 1: The graph of a function $h(x)$ and the coordinates of a point $(x_0, h(x_0))$ on the graphs of h are given below. Find $h'(x_0)$ by analyzing the graph.



$h'(3) \stackrel{?}{=}$



$h'(3) \stackrel{?}{=}$

Note: In the following problems you can use the fact that the derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$. (See the calculation carried out in Chapter 2, Example 15.)

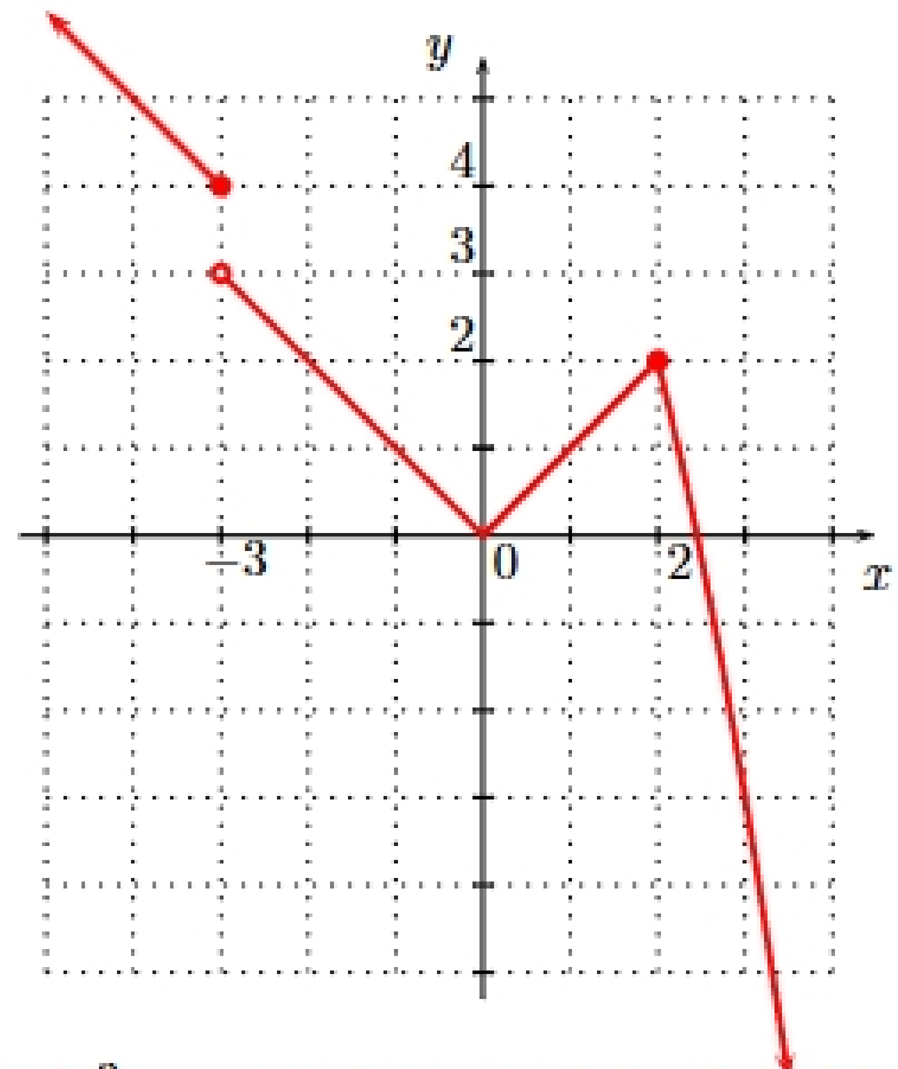
Example 2: Consider the function $f(x) = 3x^2 - 6x - 10$. Write the equation of the tangent line to the graph of f at $x = -2$ in the form $y = mx + b$, for appropriate constants m and b.

Example 3: Consider the function $g(x) = -3x^2 + 7x - 6$. Write an equation of the tangent line to the graph of g at $x = 1$. For which values of y_1 and y_2 does this tangent line go through the points $(-1, y_1)$ and $(4, y_2)$?

Example 4: Suppose that the equation of the tangent line to the graph of the function $f(x) = \sqrt{x} + a$ at $x = 16$ is given by $y = mx + 5$. Find a and m. $\left(\textbf{Hint: } \text{You may use } f'(x) = \dfrac{1}{2\sqrt{x}}.\right)$

Example 5: Determine the x values where the derivative of the function is not defined (that is the points where the function is not differentiable). Is the function continuous at those points?

$$g(x) = \begin{cases} -x+1 & \text{if } x \leq -3 \\ |x| & \text{if } -3 < x < 2 \\ -x^2+6 & \text{if } x \geq 2 \end{cases}$$



Example 6: Determine the x values where the derivative of $h(x) = |x^2 - 7x + 10|$ is not defined. Is $h(x)$ continuous at those points? (**Hint:** first draw the graph of the equation $y = x^2 - 7x + 10$ and then draw the graph of the function h.)

Next, we use the definition of the derivative to learn how to differentiate functions of the following types:

$$f(x) = (x+\alpha)^2 \qquad f(x) = \frac{1}{x+\alpha} \qquad f(x) = \sqrt{x+\alpha} \qquad f(x) = (x+\alpha)^3$$

where α is an arbitrary real number. For each type of function, the calculation of the limit has to be treated with a different technique.

Example 7: Let $f(x) = (x+4)^2$.

(a) Find constants A, B, and C such that $\dfrac{f(x+h) - f(x)}{h} = Ax + Bh + C$.

(b) Show that the derivative of f is given by the expression $f'(x) = 2x + 8 = 2(x+4)$.

(c) Find $f'(5)$. Write the equation of the tangent line to the graph of f at $x = 5$ in the form $y = mx + b$.