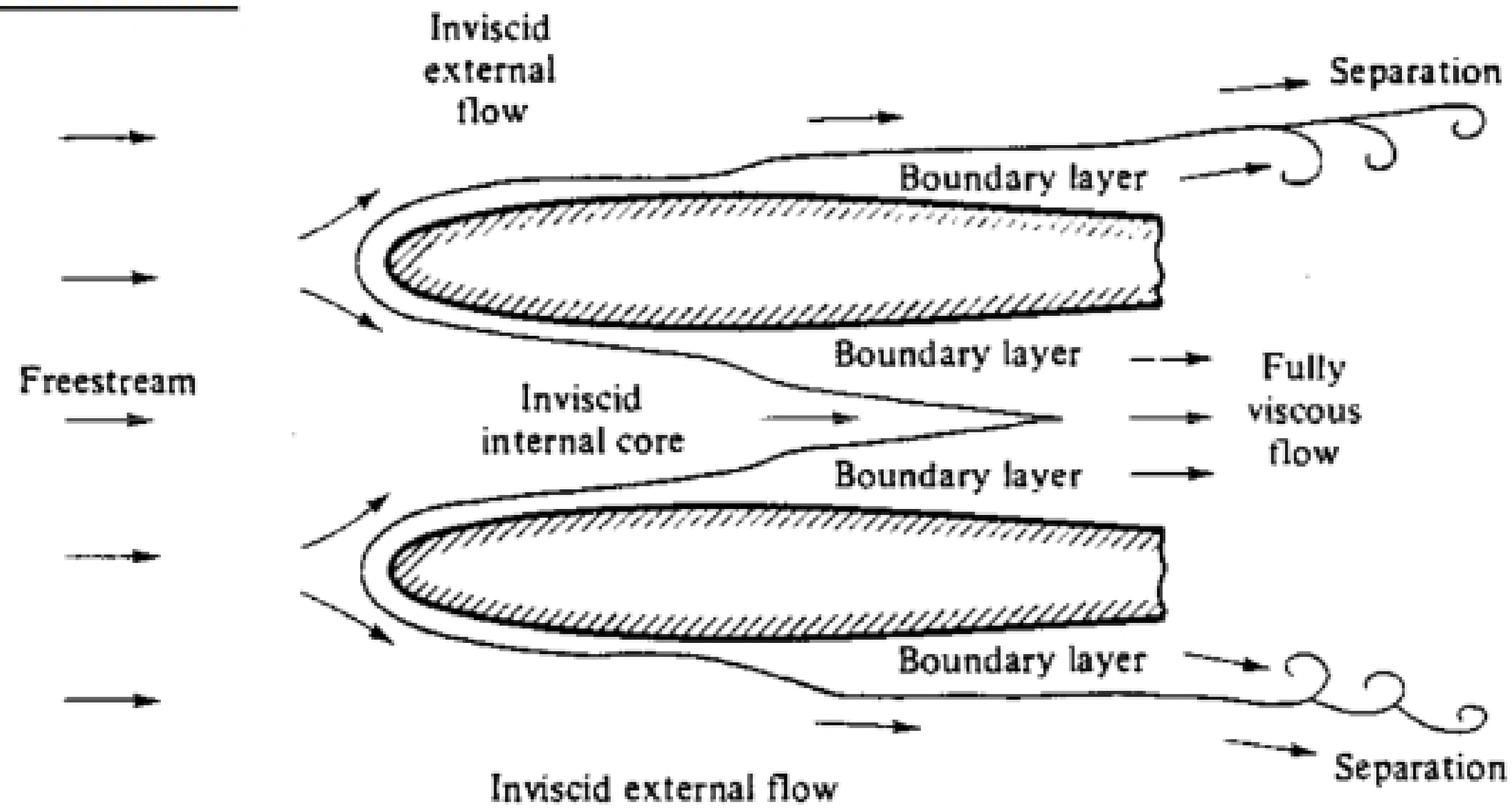
Chapter 8: Inviscid Incompressible Flow: a Useful Fantasy

8.1 Introduction



For high Re external flow about streamlined bodies viscous effects are confined to boundary layer and wake region. For regions where the B.L is thin i.e. favorable pressure gradient regions, Viscous/Inviscid interaction is weak and traditional B.L theory can be used. For regions where B.L is thick and/or the flow is separated i.e. adverse pressure gradient regions more advanced boundary layer theory must be used including viscous/Inviscid interactions.

For internal flows at high Re viscous effects are always important except near the entrance. Recall that vorticity is generated in regions with large shear. Therefore, outside the B.L and wake and if there is no upstream vorticity then $\underline{\omega}=0$ is a good approximation. Note that for compressible flow this is not the case in regions of large entropy gradient.

Also, we are neglecting noninertial effects and other mechanisms of vorticity generation.

Potential flow theory

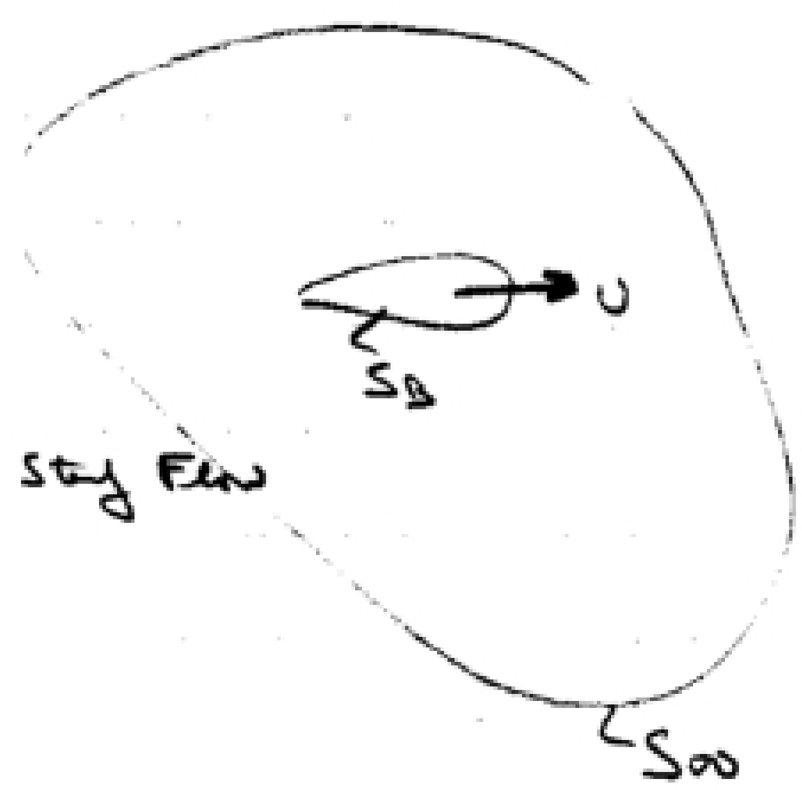
1) Determine ϕ from solution to Laplace equation

$\nabla^2\phi = 0$

B.C:

at $S_B: \underline{V}.\underline{n} = 0 \rightarrow \dfrac{\partial\phi}{\partial n} = 0$

at $S_\infty: \underline{V} = \nabla\phi$



Note:

F: Surface Function

$$\frac{DF}{Dt}=0\rightarrow\frac{\partial F}{\partial t}+\underline{V}.\nabla F=0\rightarrow\underline{V}.\underline{n}=-\frac{1}{|\nabla F|}\frac{\partial F}{\partial t}\quad \text{for steady flow } \underline{V}.\underline{n}=0$$

2) Determine $\underline{V}$ from $\underline{V}=\nabla\phi$ and p(x) from Bernoulli equation
Therefore, primarily for external flow application we now consider inviscid flow theory ($\mu=0$) and incompressible flow ($\rho=const$)

Euler equation:

$$\nabla.\underline{V}=0$$

$$\rho\frac{D\underline{V}}{Dt}=-\nabla p+\rho\underline{g}$$

$$\rho\frac{\partial\underline{V}}{\partial t}+\rho\underline{V}.\nabla\underline{V}=-\nabla(p+\gamma z)$$

$$\underline{V}.\nabla\underline{V}=\nabla\frac{V^2}{2}-\underline{V}\times\underline{\omega}$$

$$Where: \underline{\omega}=\nabla\times\underline{V}=vorticity=2\times \text{fluid angular velocity}$$

$$\Rightarrow\rho\frac{\partial\underline{V}}{\partial t}+\nabla(p+\frac{1}{2}\rho V^2+\gamma z)=\rho\underline{V}\times\underline{\omega}$$

$$If\ \ \underline{\omega}=0\ \ ie\ \ \nabla\times\underline{V}=0\ \ then\ \ \underline{V}=\nabla\phi:$$

$$\boxed{\rho\frac{\partial\phi}{\partial t}+p+\frac{1}{2}\rho\nabla\phi\cdot\nabla\phi+\gamma z=B(t)}$$

Bernoulli's Equation for unsteady incompressible flow, not *f(x)*

Continuity equation shows that GDE for ϕ is the Laplace equation which is 2nd order linear PDE ie superposition principle is valid. (Linear combination of solution is also a solution)

$$\nabla\cdot\underline{V}=\nabla\cdot\nabla\phi=\nabla^2\phi=0$$

$$\phi=\phi_1+\phi_2$$

$$\nabla^2\phi=0\Rightarrow\nabla^2(\phi_1+\phi_2)=0\Rightarrow\nabla^2\phi_1+\nabla^2\phi_2=0\Rightarrow\begin{cases}\nabla^2\phi_1=0\\ \nabla^2\phi_2=0\end{cases}$$

Techniques for solving Laplace equation:

1) superposition of elementary solution (simple geometries)
2) surface singularity method (integral equation)
3) FD or FE
4) electrical or mechanical analogs
5) Conformal mapping (for 2D flow)
6) Analytical for simple geometries (separation of variable etc)

8.2 Elementary plane-flow solutions:

Recall that for 2D we can define a stream function such that:

$$u = \psi_y$$

$$v = -\psi_x$$

$$\omega_z = v_x - u_y = \frac{\partial}{\partial x}(-\psi_x) - \frac{\partial}{\partial y}(\psi_y) = -\nabla^2\psi = 0$$

i.e. $\nabla^2\psi = 0$

Also recall that ϕ and ψ are orthogonal.

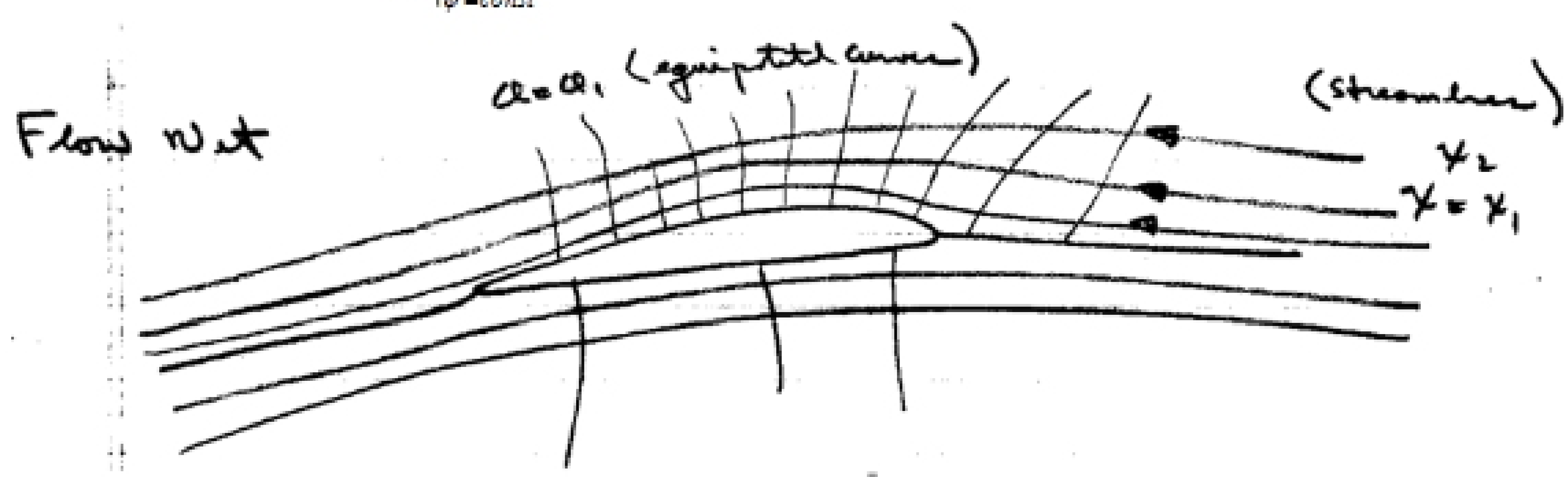
$$u = \psi_y = \phi_x$$

$$v = -\psi_x = \phi_y$$

$$d\phi = \phi_x dx + \phi_y dy = udx + vdy$$

$$d\psi = \psi_x dx + \psi_y dy = -vdx + udy$$

i.e. $\left.\frac{dy}{dx}\right|_{\phi=const} = -\frac{u}{v} = \frac{-1}{\left.\frac{dy}{dx}\right|_{\psi=const}}$



8.2 Elementary plane flow solutions

Uniform stream

$$u = U_\infty = \psi_y = \phi_x = const$$

$$v = 0 = -\psi_x = \phi_y$$

i.e. $\phi = U_\infty x$, $\psi = U_\infty y$

Note: $\nabla^2\phi = \nabla^2\psi = 0$ is satisfied.

$\underline{V} = \nabla\phi = U_\infty \hat{i}$

Say a uniform stream is at an angle α to the x-axis: