Sample Size for Comparing Two Means

This is useful for planning studies with two independent groups. Assume σ_1^2 and σ_2^2 are known. Want equal sample size in each group. Level of significance = α. Two-sided test. Power = 1 - β. The appropriate sample size for each group is

$$n = \frac{(\sigma_1^2 + \sigma_2^2)(z_{1-\alpha/2} + z_{1-\beta})^2}{\Delta^2}$$

$$\text{where } \Delta = |\mu_2 - \mu_1|$$

where μ_1 and μ_2 are the two population means
σ_1^2 and σ_2^2 are the two population variances.

Note that the sample size depends on Δ, the minimum detectable difference, the two variances, the level of significance (α), and power ($1 - \beta$).

Example: We want to test whether there is a significant difference between the mean blood-clotting times of persons using two different drugs. We wish to test for a significant difference, at the 0.05 level of significance, and with a 90% chance of detecting a true difference between population means as small as 0.5 minutes. Suppose the within-population variability, based on previous studies, is 0.52 min^2 for each population. What sample size is needed for each drug?

σ_1^2 = 0.52 min^2, σ_2^2 = 0.52 min^2, $z_{1-\alpha/2} = z_{.975}$ = 1.96, $z_{1-\beta} = z_{.90}$ = 1.28, Δ = .5 minutes

$$n = \frac{(\sigma_1^2 + \sigma_2^2)(z_{1-\alpha/2} + z_{1-\beta})^2}{\Delta^2}$$

$$= \frac{(.52 + .52)(1.96 + 1.28)^2}{.5^2}$$

$$= \frac{(1.04)(10.4976)}{.25}$$

$$= 43.67 \text{ or } 44$$

Sometimes, we expect a different sample size in each group. We may want one group to have k times the number of subjects in the other group where $k \geq 1$. Let $n_2 = k * n_1$. Therefore,

$$n_1 = \frac{\left(\sigma_1^2 + \sigma_2^2/k\right)\left(z_{1-\alpha/2} + z_{1-\beta}\right)^2}{\Delta^2}$$

$$n_2 = \frac{\left(k\sigma_1^2 + \sigma_2^2\right)\left(z_{1-\alpha/2} + z_{1-\beta}\right)^2}{\Delta^2}$$

$$\Delta = |\mu_2 - \mu_1|$$

$$k = \frac{n_2}{n_1}$$

Example: For the same example, suppose we want 2 times as many subjects in sample 2 as in sample 1.

$\sigma_1^2 = 0.52$ min^2, $\sigma_2^2 = 0.52$ min^2, $z_{1-\alpha/2} = z_{.975} = 1.96$, $z_{1-\beta} = z_{.90} = 1.28$, $\Delta = .5$ minutes, $k = 2$

$$
\begin{aligned}
n_1 &= \frac{(\sigma_1^2 + \sigma_2^2/k)(z_{1-\alpha/2} + z_{1-\beta})^2}{\Delta^2} \\
&= \frac{(.52 + .52/2)(1.96 + 1.28)^2}{.5^2} \\
&= \frac{(.78)(3.24)^2}{.25} \\
&= \frac{8.188128}{.25} \\
&= 32.75 \text{ or } 33 \\
n_2 &= 2 * 33 = 66
\end{aligned}
$$

Note: <u>If the variances of the two groups are the same, then for a given α, β, the smallest total sample size is achieved by the equal sample size rule</u>.

Note: For a one-tail test, substitute $1-\alpha$ for $1-\alpha/2$ in all above equations. This will result in smaller sample sizes.

Power

Sometimes we have a predetermined sample size and want to know how much power there is for detecting a specific alternative. Assume σ_1^2 and σ_2^2 are known. We want to conduct a two-tailed test with significance level α. We want to know how much power there will be to test the hypothesis

H_0: $\mu_1 = \mu_2$
H_1: $\mu_1 \neq \mu_2$ for the specific alternative $|\mu_1 - \mu_2| = \Delta$

$$
\text{Power} = \Phi\left[-z_{1-\alpha/2} + \frac{\sqrt{n_1}\,\Delta}{\sqrt{\sigma_1^2 + \sigma_2^2/k}}\right]
$$

or

$$
\Pr\left[z < -z_{1-\alpha/2} + \frac{\sqrt{n_1}\,\Delta}{\sqrt{\sigma_1^2 + \sigma_2^2/k}}\right]
$$

where $k = n_2/n_1$