NOTES FOR 6.002 LECTURE #14, APRIL 1, 2003

READ: 14.1–14.3

A DIFFERENT CLASS OF EXCITATIONS OF LINEAR CIRCUITS: SINUSOIDS
CONSIDER A LINEAR, PASSIVE CIRCUIT (OR SYSTEM FOR THAT MATTER) INITIALLY AT REST AND EXCITED BY SINGLE SOURCE OF THE FORM:

$$f(t) = u_{-1}(t)\,A\cos\omega t \quad \text{I.E.} \quad f(t) = \begin{cases} 0 & t<0 \\ A\cos\omega t & t>0 \end{cases}$$

HOW IS THE RESPONSE $y(t)$ – A VOLTAGE OR A CURRENT SOMEWHERE IN THE CIRCUIT – DETERMINED?

AS AN ILLUSTRATION CONSIDER A SECOND-ORDER SYSTEM DESCRIBED BY THE FOLLOWING D.E.

α, ω_0 ARE CONSTANTS $\quad \dfrac{d^2y}{dt^2} + 2\alpha\dfrac{dy}{dt} + \omega_0^2 y = u_{-1}(t)\,A\cos\omega t$

AS USUAL, THE COMPLETE SOLUTION HAS TWO COMPONENTS:

1) HOMOGENEOUS SOLUTION FITS $\dfrac{d^2y}{dt^2} + 2\alpha\dfrac{dy}{dt} + \omega_0^2 y = 0$

AND IS OF THE FORM $y_h(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$ WHERE s_1, s_2 ARE SOLUTIONS OF $s^2 + 2\alpha s + \omega_0^2 = 0$ AND K_1, K_2 ARE CONSTANTS

2) PARTICULAR SOLUTION IS THE $y_p(t)$ WHICH SATISFIES $\dfrac{d^2y_p}{dt^2} + 2\alpha\dfrac{dy_p}{dt} + \omega_0^2 y_p = A\cos\omega t$ (1)

3) COMPLETE SOLUTION IS $y(t) = y_h(t) + y_p(t)$ WHERE K_1, K_2 ARE CHOSEN TO SATISFY THE INITIAL CONDITIONS ON y AND ITS DERIVATIVE.

THE GENERAL FORM OF $y_p(t)$ FOR LINEAR, CONSTANT COEFFICIENT, ORDINARY D.E.'S IS

$$y_p(t) = B\cos(\omega t + \phi)$$ WHERE B & ϕ ARE CONSTANTS SELECTED TO MAKE THE PARTICULAR EQUATION BALANCE. ALL IT TAKES IS ALGEBRA, A GOOD LIST OF TRIGONOMETRIC IDENTITIES AND PATIENCE.

THE HOMOGENEOUS PART OF THE SOLUTION ALWAYS DECAYS TO ZERO FOR RLC CIRCUITS (THE SPECIAL CASE OF NO LOSS EXCEPTED). THUS AFTER SEVERAL TIME CONSTANTS OR AFTER THE OSCILLATION HAS PETERED OUT THE PARTICULAR SOLUTION IS ALL THAT IS LEFT.

THIS POST-TRANSIENT CONDITION IS CALLED THE SINUSOIDAL STEADY STATE (SSS)

THE DEPARTED HOMOGENEOUS SOLUTION SERVED TO ACCOMODATE THE SPECIFIED INITIAL CONDITIONS TO THE COSINE DRIVE WHICH APPEAR ABRUPTLY AT $t=0$.

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CONSIDER AN ALTERNATIVE APPROACH TO DETERMINING $y_p(t)$ WHICH USES COMPLEX TIME FUNCTIONS AND REPLACES TRIG WITH ALGEBRA.
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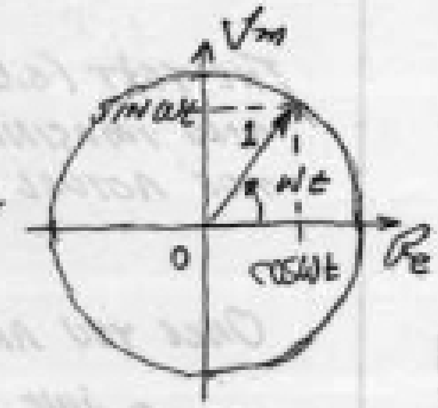
THE EULER EQUATION:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

NOTE GRAPHICAL INTERPRETATION:



$$\left|e^{j\omega t}\right| = \sqrt{\cos^2\omega t + \sin^2\omega t} = 1$$

$$\angle e^{j\omega t} = \tan^{-1}\left[\frac{\sin\omega t}{\cos\omega t}\right] = \omega t$$

$e^{j\omega t}$ IS A UNIT VECTOR IN THE COMPLEX PLANE ROTATING CCW WITH ANGULAR VELOCITY ω.

TO FIND $y_p(t)$ ASSUME: THE DRIVE $A\cos\omega t$ IS REPLACED BY $Ae^{j\omega t}$ (2)

THE RESPONSE MUST BE OF THE FORM $\bar{B}e^{j\omega t}$ WHERE $\bar{B}$ IS A CONSTANT, ALBEIT COMPLEX.

SUBSTITUTE IN THE D.E. (1): SOLVE FOR $\bar{B}$

$$(j\omega)^2\bar{B}e^{j\omega t} + (j\omega)2\alpha\bar{B}e^{j\omega t} + \omega_0^2\bar{B}e^{j\omega t} = Ae^{j\omega t}$$

$$\bar{B} = \frac{A}{(j\omega)^2 + 2\alpha(j\omega) + \omega^2} = \frac{A}{(\omega_0^2 - \omega^2) + j(2\alpha\omega)}$$

FINALLY EXPRESS THE COMPLEX FUNCTION OF ω IN POLAR FORM:

$$\bar{B} = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\alpha\omega)^2}}\, e^{-j\theta}$$ WHERE $\tan\theta = \frac{2\alpha\omega}{\omega_0^2 - \omega^2}$

SO THAT THE SSS RESPONSE IS:

$$\bar{y}_p(t) = \frac{A}{\sqrt{\quad}}\, e^{-j\theta} e^{j\omega t} = \frac{A}{\sqrt{\quad}}\, e^{j(\omega t - \theta)} \qquad (3)$$

NOTE: THE ACTUAL <u>INPUT</u> $A\cos\omega t$ IS THE <u>REAL PART</u> OF THE ASSUMED INPUT (2).

AND THE ACTUAL <u>OUTPUT</u> IS THE <u>REAL PART</u> OF (3) OR

$$y_p(t) = \frac{A}{\sqrt{\quad}}\cos(\omega t - \theta)$$

THIS ASSUMPTION OF A COMPLEX INPUT, AND DETERMINATION OF THE ACTUAL OUTPUT AS THE REAL PART OF THE ALGEBRAICALLY-DERIVED COMPLEX OUTPUT, WORKS BECAUSE THE ALGEBRA IS LINEAR AND THE OPERATION OF FINDING THE REAL PART IS LINEAR, AND <u>LINEAR OPERATIONS COMMUTE</u>: THE ORDER DOESN'T MATTER.

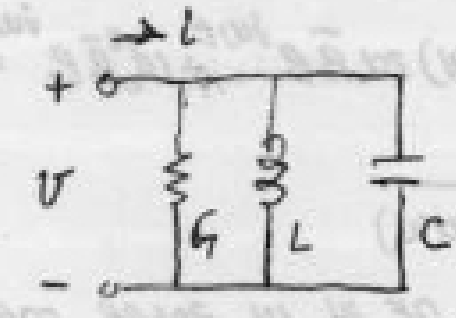
THE TEXT (6.002 NOTES) TAKES A DIFFERENT APPROACH IN WHICH THE REAL AND IMAGINARY PARTS OF $e^{j\omega t}$ ARE TREATED AS SEPARATE INPUTS, WITH THE ACTUAL OUTPUT CALCULATED BY USING SUPERPOSITION. SAME CONCLUSION

ONCE YOU ARE COMFORTABLE WITH THE FACT THAT INTEGRATION OR DIFFERENTIATION OF $e^{j\omega t}$ YIELDS A FUNCTION OF $j\omega$ IN WHICH $e^{j\omega t}$ IS A FACTOR, I.E.:

$$\frac{d}{dt}\left(\bar{A}e^{j\omega t}\right) = \bar{A}(j\omega)\,e^{j\omega t} \quad \text{AND} \quad \int \bar{A}e^{j\omega t}dt = \frac{A}{j\omega}e^{j\omega t}$$

YOU CAN DISPENSE WITH THE $e^{j\omega t}$ BECAUSE IT APPEARS IN EVERY TERM ON BOTH SIDES, AND CAN BE DIVIDED OUT.

CONSIDER A CIRCUIT EXAMPLE:



FIND $i(t)$ IN SSS

$v(t) = V\cos(\omega t + \phi)$

THE D.E. IS:

$$i = Gv + \frac{1}{L}\int v\,dt + C\frac{dv}{dt} \quad \text{(KCL)}$$

REPLACE $v(t)$ BY $Ve^{j\phi} \equiv \bar{V}$

DETERMINE $\bar{I}$ (COMPLEX FUNCTION) BY SUBSTITUTING FOR v IN THE D.E.

$$\bar{I} = G\bar{V} + \left(\frac{1}{j\omega L}\right)\bar{V} + (j\omega C)\bar{V}$$

(4) $$\bar{I} = \bar{V}\left[G + \frac{1}{j\omega L} + j\omega C\right] = \bar{V}\,\frac{j\omega LG + (1 - \omega^2 LC)}{j\omega L} = \frac{\omega LG - j(1-\omega^2 LC)}{\omega L}$$

OR, IN POLAR FORM: $\bar{I} = \bar{V}\,\dfrac{\sqrt{(\omega LG)^2 + (1-\omega^2 LC)^2}}{\omega L}e^{-j\theta}$ WHERE $\tan\theta = \dfrac{1-\omega^2 LC}{\omega L G}$

THUS: $$i(t) = \mathrm{Re}\left[V\frac{\sqrt{\quad}}{\omega L}e^{j(\phi-\theta)}e^{j\omega t}\right] = V\frac{\sqrt{\quad}}{\omega L}\cos(\omega t + \phi - \theta)$$

THE RATIO $\dfrac{\bar{I}}{\bar{V}} = \dfrac{\sqrt{\quad}}{\omega L}e^{-j\theta}$ IS THE <u>ADMITTANCE</u> $\bar{Y}$ LOOKING IN AT THE TERMINALS. ITS RECIPROCAL $\bar{Z} = \dfrac{1}{\bar{Y}}$ IS THE <u>IMPEDANCE</u> LOOKING IN.

$\bar{V}$ AND $\bar{I}$ ARE <u>COMPLEX AMPLITUDES</u>.

CONSIDER THE IMPEDANCES (AND ADMITTANCES) OF SIMPLE ELEMENTS