

3.1 We throw a fair coin 6 times. Let T be the random variable whose value is the number of tails (thus T is between 0 and 6). Let H be the random variable whose value is the number of heads. Circle the correct answer:

TRUE - FALSE: $E[T] = 3$ (i. e., the expected value of S is 3)

TRUE - FALSE: $P(H = 3) \geq 1/2$ (i. e., the probability that T has value 3 is at least $1/2$)

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TRUE - FALSE: $P(H \leq 3) \geq 1/2$ (i. e., the probability that H has value at most 3 is at least $1/2$)

$T_i =$ indicator random variable of the event
that i -th throw results in a tail

$$E[T] = E[T_1] + \dots + E[T_6] = 6 \cdot (1/2) = 3$$

$$P(T=3) = P(H=3) = \text{binomial}(6,3)/2^6 = 5/16 < 1/2$$

3.2 Let $A[1..n]$ be an array of n distinct numbers. A pair of numbers (i, j) such that $i < j$ and $A[i] > A[j]$ is called an *inversion* of A . Suppose that A is a uniformly random permutation. What is the expected number of inversions in A ?

HINT 1: Let $X_{i,j}$ be the indicator random variable of the event “ (i, j) is an inversion” (thus $X_{i,j}$ is 1 if (i, j) is an inversion, and 0 otherwise). What is the expected value $E[X_{i,j}]$ of $X_{i,j}$?

HINT 2: Let X be the random variable whose value is the number of inversions in A . Can you express X in terms of the $X_{i,j}$?

$$E[X_{i,j}] = 1/2$$

(consider only $i < j$)

$$X = \sum_{1 \leq i < j \leq n} X_{i,j}$$

$$E[X] = n(n-1)/2$$

3.3 (bonus problem) Professor Gyurcsany plays the following game: he keeps flipping a fair coin and stops when he saw more heads than tails (thus, in particular, if the first throw results in a head, professor stops the game after 1 throw). What is the expected number of throws that the professor makes?

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$$E = 1 + (1/2) * 0 + (1/2) * (T + T)$$

$$E = 1 + T$$