Lecture #22 Waves in a Cold, Unmagnetized Plasma Howes ①

I. The Cold Plasma Approximation

A. 1. MHD is the appropriate description of plasma dynamics for a ~~magnetized~~ plasma under strongly collisional, magnetized, and non-relativistic conditions. It provides the low-frequency (compared to ω_{ci}) limit of plasma behavior

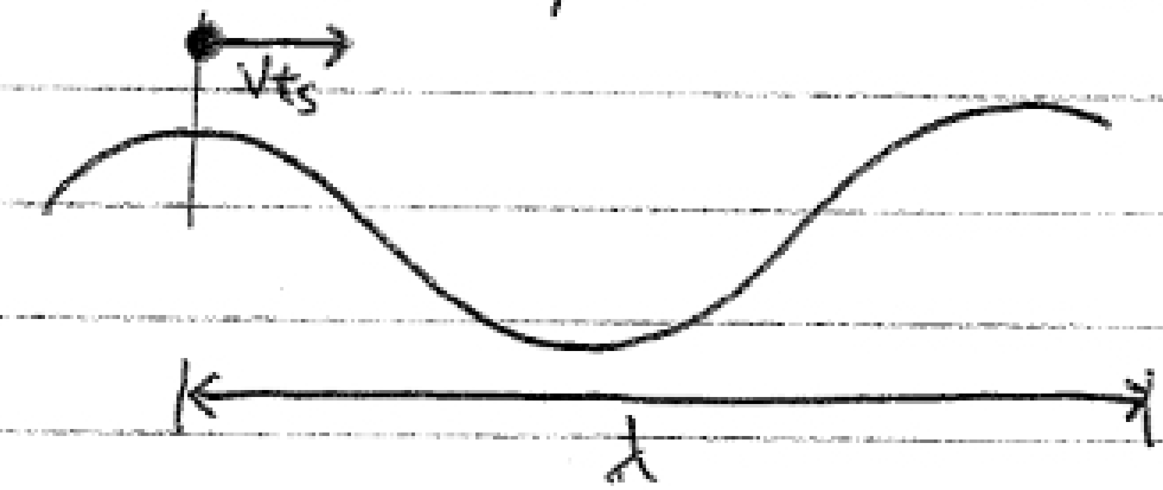
2. Many higher frequency plasma waves are well-described by the cold plasma equations.

B. When is a plasma cold?

1. The cold plasma approximation is appropriate when the thermal spread in velocities of different electrons and ions is ignored.

2. The Cold plasma description will work when the distance travelled at their thermal velocity over one wave period is small compared to the wavelength.

a.



Wave period $\tau = \frac{2\pi}{\omega}$

Using $k = \frac{2\pi}{\lambda}$

$$\frac{\begin{pmatrix}\text{Distance travelled} \\ \text{due to thermal motion}\end{pmatrix}}{\text{Wave length}} = \frac{(v_{ts})\left(\frac{2\pi}{\omega}\right)}{\lambda} \ll 1 \Rightarrow v_{ts} \ll \frac{\omega}{k} \equiv v_p$$

b. Thus, thermal velocity must be small compared to the phase velocity of the wave, v_p.

c. Since electrons have a small mass $m_e \ll m_i$, $v_{te} \gg v_{ti}$ for the same temperature $T_i = T_e$.

d. ~~The cold plasma~~ Therefore, we require

$$\boxed{v_{te} \ll \frac{\omega}{k}}$$ Cold Plasma Approximation

Lecture #22 (Continued) Howes ③

2. (Continued)

C. Two Fluid Treatment:

1. The requirement $v_{te} \ll \frac{\omega}{k}$ implies we will be looking at high-frequency phenomena.

a. At high frequencies, much of the physics occurs because of different ion and electron responses (due to $m_i \gg m_e$). Thus, we need a Two-Fluid Approach

b. High frequencies also suggest $\omega \gg \nu_{ei}$, so collisional effects may be neglected.

c. Finally, the thermal pressure term may be neglected.

II. Cold Plasma Equations

A. Neglecting viscosity, pressure, and collisional drag in the Two-Fluid Equations yields the Cold Plasma Equations:

Continuity Eq: $$\frac{\partial n_s}{\partial t} + \underset{\sim}{U_s} \cdot \nabla n_s = -n_s \nabla \cdot \underset{\sim}{U_s}$$

Momentum Eq: $$m_s n_s \left[\frac{\partial \underset{\sim}{U_s}}{\partial t} + \underset{\sim}{U_s} \cdot \nabla \underset{\sim}{U_s}\right] = q_s n_s (\underset{\sim}{E} + \underset{\sim}{U_s} \times \underset{\sim}{B})$$

Maxwell's Eq's:
$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$$
$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$
$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$
$$\nabla \cdot \underset{\sim}{B} = 0$$

where Charge Density: $\rho_q = \sum_s n_s q_s$

Current Density: $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U_s}$

1. NOTE: There is no closure problem in the Cold Plasma Equations.

a. 14 Equations for 14 unknowns: $n_i, n_e, \underset{\sim}{U_i}, \underset{\sim}{U_e}, \underset{\sim}{E}, \underset{\sim}{B}$

b. Effectively, the limit $T_s \to 0$ (Cold plasma limit) acts as the Equation of State, closing the system

III. Waves in a Cold, Unmagnetized Plasma:

Once again we Linearize, Fourier Transform, and Solve

A. Linearization:

1. Consider a single species, singly ionized plasma.
   We take a homogeneous system with no fields or flows.

   a. $n_{i0} = n_{e0}$
   b. $q_i = -q_e$
   } Quasineutrality $\rho_{q0} = \sum_s n_{s0} q_s = q_i n_{i0} + q_e n_{e0} = 0$

   But, $\rho_{q1} = q_i n_{i1} + q_e n_{e1} \neq 0$ (in general)

   c. $\underset{\sim}{U}_{i0} = 0$, $\underset{\sim}{U}_{e0} = 0$

   d. $\underset{\sim}{B}_0 = 0$, $\underset{\sim}{E}_0 = 0$

2. Take
$$n_i = n_{i0} + \epsilon n_{i1}, \quad n_e = n_{e0} + \epsilon n_{e1}$$
$$\underset{\sim}{U}_i = \epsilon \underset{\sim}{U}_{i1}, \quad \underset{\sim}{U}_e = \epsilon \underset{\sim}{U}_{e1}$$
$$\underset{\sim}{B} = \epsilon \underset{\sim}{B}_1, \quad \underset{\sim}{E} = \epsilon \underset{\sim}{E}_1$$

3. Continuity Eq:
$$\cancel{\frac{\partial n_{i0}}{\partial t}}^{\,0} + \epsilon \frac{\partial n_{i1}}{\partial t} + \epsilon \underset{\sim}{U}_{i1} \cdot \overset{\text{uniform}}{\cancel{\nabla n_{i0}}}^{\,0} + \epsilon^2 \underset{\sim}{U}_{i1} \cdot \nabla n_{i1} = -\epsilon n_{i0} \nabla \cdot \underset{\sim}{U}_{i1} + \epsilon^2 n_{i1} \nabla \cdot \underset{\sim}{U}_{i1}$$

$$O(\epsilon) \Rightarrow \boxed{\frac{\partial n_{i1}}{\partial t} = -n_{i0} \nabla \cdot \underset{\sim}{U}_{i1}} \quad \text{Similarly} \quad \boxed{\frac{\partial n_{e1}}{\partial t} = -n_{e0} \nabla \cdot \underset{\sim}{U}_{e1}}$$

4. Momentum Eq:
$$\epsilon m_i n_{i0} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} + \epsilon^2 m_i n_{i1} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} + \epsilon^2 m_i n_{i0} \underset{\sim}{U}_{i1} \cdot \nabla \underset{\sim}{U}_{i1} + \epsilon^3 m_i n_{i1} \underset{\sim}{U}_{i1} \cdot \nabla \underset{\sim}{U}_{i1}$$
$$= \epsilon q_i n_{i0} \underset{\sim}{E}_1 + \epsilon^2 q_i n_{i1} \underset{\sim}{E}_1 + \epsilon^2 q_i n_{i0} \underset{\sim}{U}_{i1} \times \underset{\sim}{B}_1 + \epsilon^3 q_i n_{i1} \underset{\sim}{U}_{i1} \times \underset{\sim}{B}_1$$

$$O(\epsilon): \quad m_i n_{i0} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} = q_i n_{i0} \underset{\sim}{E}_1 \Rightarrow \boxed{\frac{\partial \underset{\sim}{U}_{i1}}{\partial t} = \frac{q_i}{m_i} \underset{\sim}{E}_1} \text{ and } \boxed{\frac{\partial \underset{\sim}{U}_{e1}}{\partial t} = \frac{q_e}{m_e} \underset{\sim}{E}_1}$$