Combinatorics

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Notes

Combinatorics I

Introduction

Combinatorics is the study of collections of objects. Specifically, *counting* objects, arrangement, derangement, etc. of objects along with their mathematical properties.

Counting objects is important in order to analyze algorithms and compute discrete probabilities.

Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability.

Notes

Combinatorics II

Introduction

A simple example: How many arrangements are there of a deck of 52 cards?

In addition, combinatorics can be used as a proof technique.

A *combinatorial proof* is a proof method that uses counting arguments to prove a statement.

Notes

Product Rule

If two events are not mutually exclusive (that is, we do them separately), then we apply the product rule.

Theorem (Product Rule)

Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second, then there are

$$n_1 \cdot n_2$$

ways of doing the overall procedure.

Notes

Sum Rule I

If two events are mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule.

Theorem (Sum Rule)

If an event e_1 can be done in n_1 ways and an event e_2 can be done in n_2 ways and e_1 and e_2 are mutually exclusive, then the number of ways of both events occurring is

$$n_1 + n_2$$

Notes

Sum Rule II

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur is

$$n_1 + n_2 + \cdots n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be pairwise *disjoint* sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

In fact, this is a special case of the general *Principle of Inclusion-Exclusion*.

Notes

Principle of Inclusion-Exclusion (PIE) I

Introduction

Say there are two events, e_1 and e_2 for which there are n_1 and n_2 possible outcomes respectively.

Now, say that only *one* event can occur, not both.

In this situation, we cannot apply the sum rule? Why?

Notes

Principle of Inclusion-Exclusion (PIE) II

Introduction

We cannot use the sum rule because we would be *over counting* the number of possible outcomes.

Instead, we have to count the number of possible outcomes of e_1 and e_2 *minus* the number of possible outcomes in common to both; i.e. the number of ways to do both "tasks".

If again we think of them as sets, we have

$$|A_1| + |A_2| - |A_1 \cap A_2|$$

Notes

Principle of Inclusion-Exclusion (PIE) III

Introduction

More generally, we have the following.

Lemma

Let A, B be subsets of a finite set U. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cap B| \leq \min\{|A|, |B|\}$
3. $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
4. $|\overline{A}| = |U| - |A|$
5. $|A \oplus B| = |A \cup B| - |A \cap B| = A + B - 2|A \cap B| = |A \setminus B| + |B \setminus A|$
6. $|A \times B| = |A| \times |B|$

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