COMPLEX NUMBERS

1.1 Complex Numbers

The concept of imaginary numbers occurs early in the discussion of algebraic equations. Thus if we solve the quadratic equation

$$x^2 + 1 = 0$$

we find the solutions $x = \pm i$, where $i = \sqrt{-1}$. More generally we find hybrid forms, called complex numbers, containing both real and imaginary parts. If, for example,

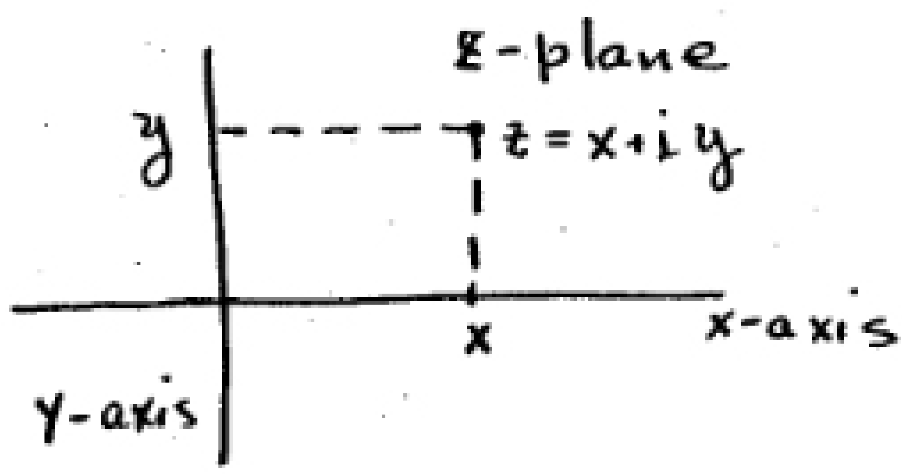
$$x^2 - 2x + 2 = 0$$

then

$$x = 1 \pm i$$

are the solutions. The complex numbers, which extend the (real) number system, are made necessary by the solution of algebraic equations, the coefficients of which are real. It is perhaps remarkable that algebraic equations with complex coefficients have solutions which are complex - no further extension is necessary.

Complex numbers can be viewed as belonging to a two-space called the complex plane



According to common convention a complex number is denoted by the letter z,

$$z = x+iy$$

One also defines the real and imaginary parts through

$$(1.1) \qquad x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

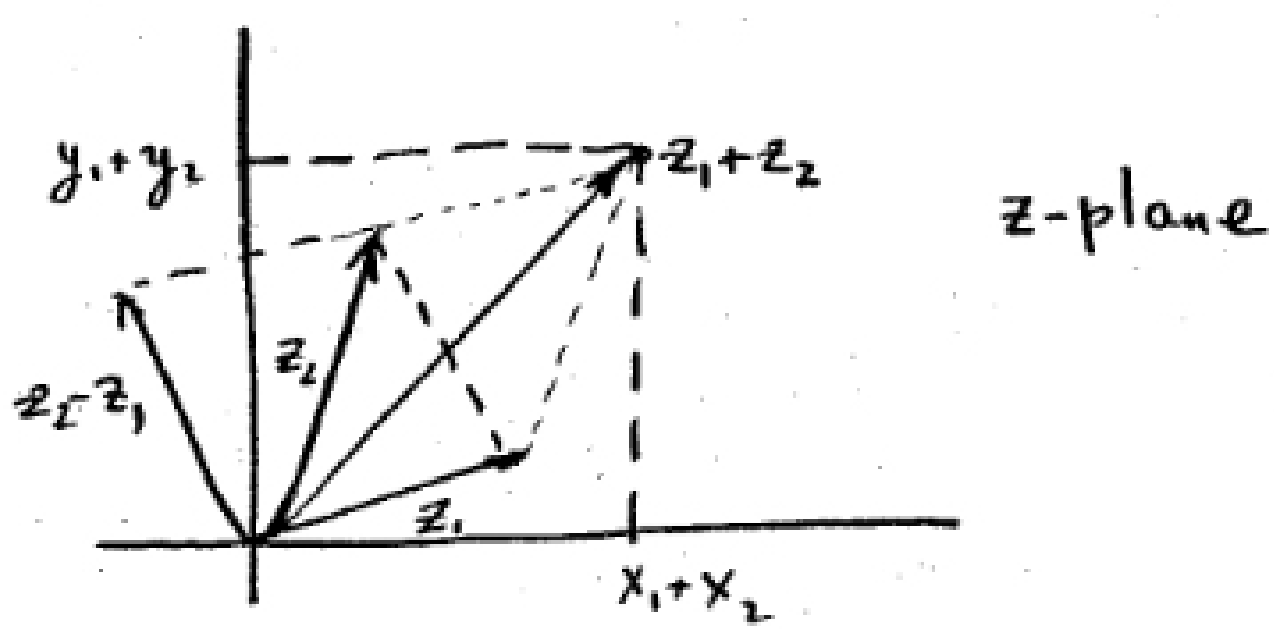
The addition of the two complex numbers

$$z_1 = x_1+iy_1, \quad z_2 = x_2+iy_2$$

follows the rules of vector addition in two spaces,

$$z = z_1+z_2 = x_1+x_2+i(y_1+y_2) = z_2+z_1$$

Both this operation and that of subtraction are indicated in the following



The figure is familiar from analytical geometry and requires no further explanation.

The complex plane differs from Cartesian two-space in that multiplication of elements is defined. In particular

$$z_1z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2) + i(x_1y_2+x_2y_1) = z_2z_1$$

where $i^2 = -1$ has been used. Actually the explicit appearance of i can be avoided by writing

$$z = (x,y)$$

and defining

$$z_1+z_2 = (x_1+x_2,y_1+y_2)$$

$$z_1z_2 = (x_1x_2-y_1y_2,x_1y_2+x_2y_1).$$

These forms are used to generate a complex arithmetic for computer use.

As usual, division is the operation inverse to that of multiplication. Thus z is called the quotient of a and b if bz = a. If we write $a = a_1+ia_2$, $b = b_1+ib_2$, and $z = x+iy$, then

$$bz = (b_1+ib_2)(x+iy) = (b_1x-b_2y) + i(b_2x+b_1y) = a_1+ia_2$$

It follows from our construction of complex numbers that in an equation the real and imaginary parts are separately equal. Thus we are led to two linear equations in x and y which when solved yields,

$$x+iy = \frac{a_1b_1+a_2b_2}{b_1^2+b_2^2} + i\,\frac{-a_1b_2+a_2b_1}{b_1^2+b_2^2}\ .$$

The denominator $b_1^2+b_2^2$ is the squared distance of the complex number b_1+ib_2 from the origin. More generally if $z=x+iy$, its