

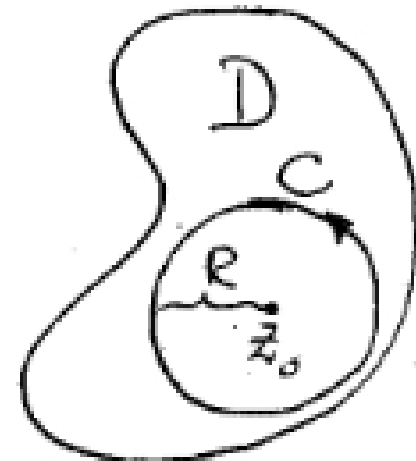
3.5. The Taylor and Laurent Expansions

Thus far we have seen analytic functions represented as power series (3.10) and also in integral form by means of the Cauchy integral theorem. Since a convergent power series is analytic, we are guaranteed an integral representation for it, but we have yet to see that an arbitrary analytic function possesses a power series representation. To show this we start with Cauchy's integral formula (3.15),

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{\zeta - z}$$

with the path a circle of radius R say, centered at z_0 and such that $f(z)$ is analytic in and on it. Then notice

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{(\zeta - z_0) \left(1 - \frac{z - z_0}{\zeta - z_0}\right)} \\ (3.19) \quad &= \frac{1}{\zeta - z_0} \left[1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \dots\right] \\ &= \frac{1}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \end{aligned}$$



which is convergent for $\left|\frac{z - z_0}{\zeta - z_0}\right| < 1$. If we multiply by $\frac{f(\zeta)}{2\pi i}$,

$$\frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z} = \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

then we obtain for $|z - z_0| < R_1 < R$, a uniformly convergent series of continuous functions, in ζ , (z and z_0 to be regarded as fixed) and the series may be integrated term by term (the proof mimics that given on page 40 and is not worth repeating).

We have therefore demonstrated that

$$\begin{aligned}
 (3.20) \quad f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-z)} d\zeta = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)^{n+1}} \\
 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0) (z-z_0)^n}{n!}
 \end{aligned}$$

where in the last step we have made use of the representation of derivatives, (3.17). This series identifies the coefficients in a power series with the derivatives of the function at the center of the circle of convergence, and what we have is the infinite Taylor expansion. (The finite form with remainder is left as an exercise.)

From the derivation of (3.20) we also learn over what radius the expansion is valid and what in fact brings this convergence to a halt. For if R is the radius of convergence of the Taylor expansion (3.20) then $|z-z_0| = R$ must contain at least one singularity of the defining function f . For if this were not true, i.e. $f(z)$ is analytic on $|z-z_0| = R$, then it would also be analytic in a neighborhood of this circle, say for $|z-z_0| \leq R+\epsilon$, $\epsilon > 0$. But this is a contradiction since the above derivation of (3.20) would then give $R+\epsilon$ as the radius of convergence. Therefore the radius convergence of an analytic function, f , about a point z_0 is the distance from z_0 to the nearest singularity of f . This certainly explains why entire functions have power series representations everywhere convergent in the plane.

We return to the simple example

$$(3.21) \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

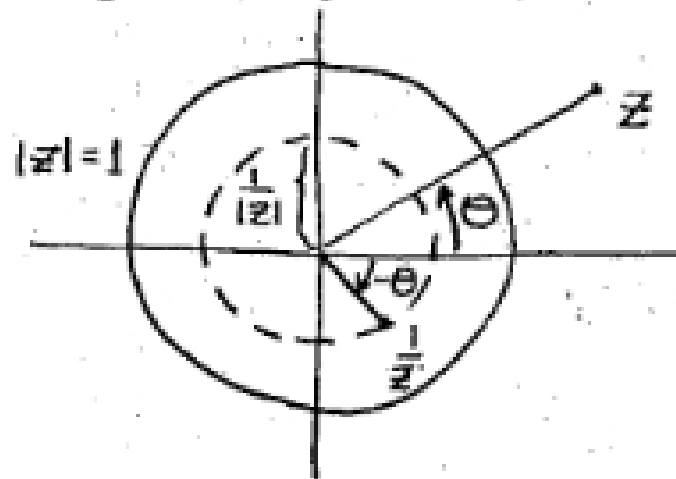
which converges for $|z| < 1$ and which certainly illustrates the above rule about the radius of convergence. Consider the transformation

$$z = \frac{1}{z}$$

which if we write $z = re^{i\theta}$ states

$$|z| = \frac{1}{r}, \quad \arg z = -\arg z = -\theta$$

It clearly maps the interior of the unit circle $|z| < 1$ into the exterior of unit circle $|z| > 1$ and vice versa. The mapping is made more precise by the figure.



If we substitute this transformation into $1/(1-z)$ we obtain

$$= \frac{z}{1-z}$$

which if expanded for $|z| < 1$ gives the convergent power series

$$= \sum_{n=1}^{\infty} z^n, \quad |z| < 1$$

and which if transformed back to the z -plane states

$$(3.22) \quad \frac{1}{1-z} = - \sum_{n=1}^{\infty} \frac{1}{z^n}, \quad |z| > 1$$

Equation (3.21) expresses the function $1/(1-z)$ inside the unit circle $|z| < 1$ by a convergent series in non-negative powers.