

III.1 Simplicial Complexes

There are many ways to represent a topological space, one being a collection of simplices that are glued to each other in a structured manner. Such a collection can easily grow large but all its elements are simple. This is not so convenient for hand-calculations but close to ideal for computer implementations. In this book, we use simplicial complexes as the primary representation of topology.

Simplices. Let u_0, u_1, \dots, u_k be points in \mathbb{R}^d . A point $x = \sum_{i=0}^k \lambda_i u_i$ is an *affine combination* of the u_i if the λ_i sum to 1. The *affine hull* is the set of affine combinations. It is a *k-plane* if the $k + 1$ points are *affinely independent* by which we mean that any two affine combinations, $x = \sum \lambda_i u_i$ and $y = \sum \mu_i u_i$, are the same iff $\lambda_i = \mu_i$ for all i . The $k + 1$ points are affinely independent iff the k vectors $u_i - u_0$, for $1 \leq i \leq k$, are linearly independent. In \mathbb{R}^d we can have at most d linearly independent vectors and therefore at most $d + 1$ affinely independent points.

An affine combination $x = \sum \lambda_i u_i$ is a *convex combination* if all λ_i are non-negative. The *convex hull* is the set of convex combinations. A *k-simplex* is the convex hull of $k + 1$ affinely independent points, $\sigma = \text{conv} \{u_0, u_1, \dots, u_k\}$. We sometimes say the u_i *span* σ . Its *dimension* is $\dim \sigma = k$. We use special names of the first few dimensions, *vertex* for 0-simplex, *edge* for 1-simplex, *triangle* for 2-simplex, and *tetrahedron* for 3-simplex; see Figure III.1. Any subset of

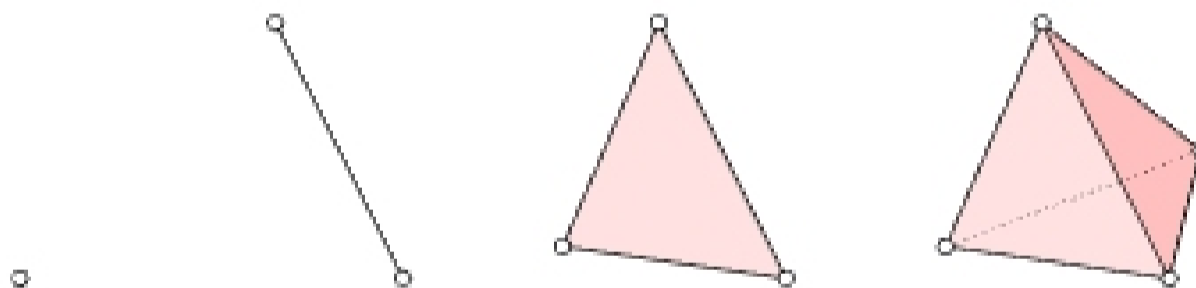


Figure III.1: From left to right: a vertex, an edge, a triangle, and a tetrahedron.

affinely independent points is again affinely independent and therefore also defines a simplex. A *face* of σ is the convex hull of a non-empty subset of the u_i and it is *proper* if the subset is not the entire set. We sometimes write $\tau \leq \sigma$ if τ is a face and $\tau < \sigma$ if it is a proper face of σ . Since a set of size $k + 1$ has 2^{k+1} subsets, including the empty set, σ has $2^{k+1} - 1$ faces, all of which are proper except for σ itself. The *boundary* of σ , denoted as $\text{bd } \sigma$, is the union of all proper faces, and the *interior* is everything else, $\text{int } \sigma = \sigma - \text{bd } \sigma$. A point $x \in \sigma$ belongs to $\text{int } \sigma$ iff all its coefficients λ_i are positive. It follows that every

point $x \in \sigma$ belongs to the interior of exactly one face, namely the one spanned by the points u_i that correspond to positive coefficients λ_i .

Simplicial complexes. We are interested in sets of simplices that are closed under taking faces and that have no improper intersections.

DEFINITION. A *simplicial complex* is a finite collection of simplices K such that $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$, and $\sigma, \sigma_0 \in K$ implies $\sigma \cap \sigma_0$ is either empty or a face of both.

The *dimension* of K is the maximum dimension of any of its simplices. The *underlying space*, denoted as $|K|$, is the union of its simplices together with the topology inherited from \mathbb{R}^d . A *polyhedron* is the underlying space of a simplicial complex. A *triangulation* of a topological space \mathbb{X} is a simplicial complex K together with a homeomorphism between \mathbb{X} and $|K|$. The topological space is *triangulable* if it has a triangulation. A *subcomplex* of K is a simplicial complex $L \subseteq K$. It is *full* if it contains all simplices in K spanned by vertices in L . A particular subcomplex is the *j -skeleton* consisting of all simplices of dimension j or less, $K^{(j)} = \{\sigma \in K \mid \dim \sigma \leq j\}$. The 0-skeleton is also referred to as the *vertex set*, $\text{Vert } K = K^{(0)}$. Skeleta are generally not full. A subset of a simplicial complex useful in talking about local neighborhoods is the *star* of a simplex τ consisting of all simplices that have τ as a face, $\text{St } \tau = \{\sigma \in K \mid \tau \leq \sigma\}$. Generally, the star is not closed under taking faces. We can make it into a complex by adding all missing faces. The result is the *closed star*, $\overline{\text{St}} \tau$, which is the smallest subcomplex that contains the star. The *link* consists of all simplices in the closed star that are disjoint from τ , $\text{Lk } \tau = \{v \in \overline{\text{St}} \tau \mid v \cap \tau = \emptyset\}$. If τ is a vertex then the link is just the difference between the closed star and the star. More generally, it is the closed star minus the stars of all faces of τ . For example if K triangulates a 2-manifold without boundary then the link of an edge is a pair of points, a 0-sphere, and the link of a vertex is a cycle of edges and vertices, a 1-sphere.

Abstract simplicial complex. It is often easier to construct a complex abstractly and to worry about how to put it into Euclidean space later.

DEFINITION. An *abstract simplicial complex* is a finite collection of sets A such that $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$.

The sets in A are its *simplices*. The *dimension* of a simplex is $\dim \alpha = \text{card } \alpha - 1$ and the dimension of the complex is the maximum dimension of any of its

simplices. A *face* of α is a non-empty subset $\beta \subseteq \alpha$, which is *proper* if $\beta \neq \alpha$. The *vertex set* is the union of all simplices, $\text{Vert } A = \bigcup A = \bigcup_{\alpha \in A} \alpha$. A *subcomplex* is an abstract simplicial complex $B \subseteq A$. Two abstract simplicial complexes are *isomorphic* if there is a bijection $b : \text{Vert } A \rightarrow \text{Vert } B$ such that $\alpha \in A$ iff $b(\alpha) \in B$. The largest abstract simplicial complex with a vertex set of size n has cardinality $2^n - 1$. Given a (geometric) simplicial complex K , we can construct an abstract simplicial complex A by throwing away all simplices and retaining only their sets of vertices. We call A a *vertex scheme* of K . Symmetrically, we call K a *geometric realization* of A but also of every abstract simplicial complex isomorphic to A . Constructing geometric realizations is surprisingly easy if the dimension of the ambient space is sufficiently high.

GEOMETRIC REALIZATION THEOREM. An abstract simplicial complex of dimension d has a geometric realization in \mathbb{R}^{2d+1} .

PROOF. Let $f : \text{Vert } A \rightarrow \mathbb{R}^{2d+1}$ be an injection whose image is a set of points in general position. Specifically, any $2d+2$ or fewer of the points are affinely independent. Let α and α_0 be simplices in A with $k = \dim \alpha$ and $k_0 = \dim \alpha_0$. The union of the two has size $\text{card}(\alpha \cup \alpha_0) = \text{card } \alpha + \text{card } \alpha_0 - \text{card}(\alpha \cap \alpha_0) \leq k + k_0 + 2 \leq 2d + 2$. The points in $\alpha \cup \alpha_0$ are therefore affinely independent, which implies that every convex combination x of points in $\alpha \cup \alpha_0$ is unique. Hence x belongs to $\sigma = \text{conv } f(\alpha)$ as well as to $\sigma_0 = \text{conv } f(\alpha_0)$ iff x is a convex combination of $\alpha \cap \alpha_0$. This implies that the intersection of σ and σ_0 is either empty or the simplex $\text{conv } f(\alpha \cap \alpha_0)$, as required. \square

Simplicial maps. Let K be a simplicial complex with vertices u_0, u_1, \dots, u_n . Every point $x \in |K|$ belongs to the interior of exactly one simplex in K . Letting $\sigma = \text{conv}\{u_0, u_1, \dots, u_k\}$ be this simplex, we have $x = \sum_{i=0}^k \lambda_i u_i$ with $\sum_{i=0}^k \lambda_i = 1$ and $\lambda_i > 0$ for all i . Setting $b_i(x) = \lambda_i$ for $0 \leq i \leq k$ and $b_i(x) = 0$ for $k+1 \leq i \leq n$ we have $x = \sum_{i=0}^n b_i(x) u_i$ and we call the $b_i(x)$ the *barycentric coordinates* of x in K . We use barycentric coordinates to construct continuous maps.

DEFINITION. A *vertex map* is a function $\varphi : \text{Vert } K \rightarrow \text{Vert } L$ with the property that the vertices of every simplex in K map to vertices of a simplex in L . Then φ can be extended to a continuous map $f : |K| \rightarrow |L|$ defined by

$$f(x) = \sum_{i=0}^n b_i(x) \varphi(u_i),$$

the *simplicial map* induced by φ .