Math 128

Final Examination – December 12, 2008

Name ______________________

8 problems, 100 points.

Instructions: Show all work – partial credit will be given, and "Answers without work are worth credit without points." You don't have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have up to four 3x5 cards, but no other notes.

1. (a) *(6 points) Calculate f_{xy}, where $f(x,y) = 3e^{-2xy}$.*
   $f_x = 3 \cdot (-2y) \cdot e^{-2xy} = -6ye^{-2xy}$, so
   $$f_{xy} = -6y \cdot (-2x) \cdot e^{-2xy} - 6e^{-2xy} = 12xye^{-2xy} - 6e^{-2xy}.$$

   (b) *(7 points) Evaluate the double integral $\int_0^1 \int_{x^2}^{\sqrt{x}} xy\,dydx$.*
   $$\int_0^1 \left[x \cdot \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 x^2 - x^5\,dx$$
   $$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1$$
   $$= \frac{1}{2}(\frac{1}{3} - \frac{1}{6}) = \frac{1}{12}.$$

   (c) *(7 points) Solve the differential equation $y' = -xy$.*
   We separate variables:
   $$\frac{dy}{dx} = -xy \implies \frac{1}{y}\,dy = -x\,dx.$$
   We then integrate:
   $$\int \frac{1}{y}\,dy = \int -x\,dx$$
   $$\ln|y| = -\frac{1}{2}x^2 + C$$
   $$|y| = e^{-\frac{1}{2}x^2 + C}$$
   $$y = \pm e^C e^{-\frac{1}{2}x^2}.$$

If we wanted, we could rewrite $A = \pm e^C$, so that $y = Ae^{-\frac{1}{2}x^2}$.

2. *Let X be a normally distributed random variable, with expected value 3 and variance 4.*

   (a) *(3 points) Set up an integral representing* $\Pr(2 \le X \le 5)$.

$$\int_2^5 \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-3}{2})^2}\, dx.$$

   (b) *(4 points) Show how to use integration by substitution to evaluate your integral from part (a). Use of the table of areas under the standard normal distribution may be helpful.*
   We take $u = \dfrac{x-3}{2}$, so that $du = \dfrac{1}{2}dx$. Then

$$\begin{aligned}\int_2^5 \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-3}{2})^2}\, dx &= \int_{u(2)}^{u(5)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= \int_{-0.5}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= A(1) - A(-0.5) = A(1) + A(0.5).\end{aligned}$$

   You can look up the values of $A(1)$ and $A(0.5)$ on the included table.

3. (a) *(8 points) Let $f(x) = e^x \sin x$. Find an upper bound A for $|f''(x)| = \left|\frac{d^2}{dx^2} e^x \sin x\right|$ on the interval $[-1, 2]$.*
   We first calculate f'':

$$\begin{aligned}f'(x) &= e^x \sin x + e^x \cos x \\ f''(x) &= (e^x \sin x + e^x \cos x) + (e^x \cos x - e^x \sin x) \\ &= 2e^x \cos x.\end{aligned}$$

   Then $|f''(x)| = 2|e^x||\cos x|$, and since $0 \le e^x \le e^2$ on the given interval, and $|\cos x| \le 1$ for all x, we find that $A = 2e^2$ is an upper bound.

   (b) *(4 points) Using your result from part (a), find an n so that the midpoint rule approximation M_n is accurate to 0.01 for the integral*

$$\int_{-1}^{2} e^x \sin x \, dx.$$

We use the error bound formula:

$$\text{error} M_n \leq \frac{2e^2 \cdot (2-(-1))^3}{24n^2} = \frac{e^2 \cdot 27}{24n^2} = \frac{e^2 \cdot 9}{8n^2}.$$

So if

$$\frac{e^2 \cdot 9}{8n^2} \leq 0.01 = \frac{1}{100},$$

then we have the desired accuracy. This means that

$$\frac{e^2 \cdot 9 \cdot 100}{8} \leq n^2, \quad \text{or} \quad \frac{30e}{2\sqrt{2}} = \frac{15e}{\sqrt{2}} \leq n.$$

Any n greater than $15e/\sqrt{2}$ has small enough error, for example $n = 30$.

4. *The random variable X has outcomes between 0 and π. The probability density function of X is $k \sin x$ for some constant k.*

   (a) *(7 points) Find k.*
   Since $k \sin x$ is a probability density function, and the outcomes of X are on $[0, \pi]$, we have that

   $$\int_0^{\pi} k \sin x \, dx = k \left[-\cos x\right]_0^{\pi} = 2k = 1.$$

   Thus $k = \frac{1}{2}$.

   (b) *(3 points) Set up an integral for $E(X)$.*
   (If you have trouble solving part (a), it's ok to leave your answer in terms of k.)

   $$E(X) = \int_0^{\pi} x \cdot \frac{1}{2} \sin x \, dx.$$

   (c) *(6 points) Evaluate your integral from part (b) to calculate $E(X)$.*
   We use integration by parts. The rule of thumb LATE tells us to