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Theory of Computability

Hierarchy theorems

Section 9.1

Space constructibility

Definition 9.1 A function **f: N→N**, where **f(n)** is at least **O(log n)**, is called ***space constructible*** if the function that maps any string **w** of length **n** (equivalently, the string 1^n) to the binary representation of the number **f(n)** is computable in space **O(f(n))**.

Intuition: Assume a machine **M** runs in **f(n)** space, and **f(n)** is space constructible. Then, for any input **w** of size **n**, using only **O(f(n))** space, we can not only tell whether **M** accepts **w**, but can also compute the value of **f(n)** itself, i.e. the amount of space used by **M** on input **w**.

Most natural functions are space constructible. Those that are not are "pathological" cases --- cases where computing the space bound is so expensive that its space cost exceeds the bound itself. This is like if counting money was more expensive than the amount of money itself.

Motivation: When **f(n)** is space constructible, we can easily construct machines for whatever purposes that control their own space consumption and make sure that it does not exceed **f(n)**.

Space hierarchy theorem

Theorem 9.3 For any space constructible function **f: N→N**, a language **A** exists that is decidable in **O(f(n))** space but not in **o(f(n))** space.

Proof. Such a language **A** is the one decided by the following algorithm (TM):

D = "On input **w**:

1. Let **n** be the length of **w**.
2. Compute **f(n)** using space constructibility, and mark off this much tape. If later stages ever attempt to use more, *reject*.
3. If **w** is not of the form **<M>10*** for some TM **M**, *reject*.
4. Simulate **M** on **w** while counting the number of steps used in the simulation. If the count ever exceeds $2^{f(n)}$, *reject*.
5. If **M** accepts, *reject*. If **M** rejects, *accept*."

Step 4 guarantees that **D** is indeed a decider (why?).

Step 2 guarantees that **D** runs in space **O(f(n))** (why?).

Step 5 guarantees that **A** is different from the language decided by any **o(f(n))** space machine (why?).