Section 2.3 Computing limits

This is an outline of some ways to evaluate $\lim_{x \to a} f(x)$ or to determine it does not exist.

Case 1: *f(x)* is given by an algebraic rule and *f(a)* is defined then *f(a)* is the limit. In other words, if substituting *a* for *x* gives a meaningful answer then that is the limit.
Examples:

$$\lim_{x \to 0} \frac{x-1}{x+2} = -\frac{1}{2} \qquad \lim_{x \to 2} (x^2 + 3x) = 10 \qquad \lim_{x \to -3} \frac{x+3}{x} = 0$$

Examples of non-meaningful expressions are $\frac{0}{0}$, $\frac{1}{0}$, 0^0 and there are others that we will discuss if they come up.

Substitution gives $\frac{nonzero}{0}$. In this case the limit does not exist as a number, the magnitude of the function goes to infinity.

If Substitution gives $\frac{0}{0}$. This is indeterminate, the limit could be any number or not exist. Do Algebra.

Case 2: Case 2: *f(x)* is a rational function, $\frac{p(x)}{q(x)}$ and *q(a)* is 0 but *p(a)* is not 0. Then substitution gives $\frac{nonzero}{0}$ so the limit does not exist.

Case 3: *f(x)* is a rational function, $\frac{p(x)}{q(x)}$, and substitution gives $\frac{0}{0}$.

This means both *p(a)* and *q(a)* have *(x-a)* as a factor. Factor the numerator and denominator and cancel common factors. Substitute again and re-evaluate it.

Example: $\lim_{x \to 3} \frac{x^2-9}{x^2-x-6}$ Substitution of 3 for *x* gives $\frac{0}{0}$.

$\frac{x^2-9}{x^2-x-6} = \frac{(x-3)(x+3)}{(x-3)(x+2)} = \frac{x+3}{x+2}$ *if* $x \neq 3$. This is OK since we do not let *x* equal 3 in taking the limit. Now substitution gives $\frac{6}{5}$. $\lim_{x \to 3} f(x) = \frac{6}{5}$.

Example: $\lim_{x \to -2} \frac{x^2+2x}{x^2+4x+4}$ Substitution of -2 for *x* gives $\frac{0}{0}$.

$\frac{x^2+2x}{x^2+4x+4} = \frac{x(x+2)}{(x+2)^2} = \frac{x}{x+2}$ *if* $x \neq -2$ Now substituting *x*= - 2 gives $\frac{-2}{0}$, so the limit does not exist.

Case 4: Conjugating is needed to simplify. Recall that $c^2 - d^2 = (c-d)(c+d)$ so also $x - a = (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})$ and $\sqrt{x} - \sqrt{a} = \dfrac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} = \dfrac{x - a}{\sqrt{x} + \sqrt{a}}$. This is called conjugating.

Example: Find $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$ Substitution gives $\frac{0}{0}$, so we conjugate and cancel.

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

Case 5: The above do not fit. Look at the graph or try to use the squeeze theorem for some possible strategies. Find $\lim_{x \to 0} \sin(\frac{1}{x})$ if it exists. Look at the graph and you will see that the function oscillates between -1 and 1 and takes on all values in between no matter how close we get to *x*=0. The limit does not exist.

Find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ *so* $-x \le \sin\left(\frac{1}{x}\right) \le x$ -x and x both approach 0 so the limit is 0. (The function is squeezed between two functions which have the same limit.)