General Physics - E&M (PHY 1308) Lecture Notes

Lecture 007: Applications of Gauss's Law and Conductors

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no tags

Goals

- Learn to work problems involving Gauss's Law

Working with Gauss's Law

As in all problems in physics, all the work is in setting up the question. If you can narrow down on what's being asked, and how to translate that into the language of mathematics (sprinkling in some physics thinking for good measure), then you will be able to setup and solve hard problems. I'll work two problems today to illustrate this.

Applying Gauss's Law: A Uniformly Charged Sphere

Imagine that you construct a spherical structure of charge, where the charge is uniformly distributed throughout the volume, whose radius is R. Can you find the electric field at all points?

To attack this problem, we can first consider the geometry. We have some charge Q spread uniformly throughout this sphere. That means we have a *constant volume charge density*, ρ. Independent of the sub-volume, dV, of the sphere we consider, $\rho = dq/dV$ will be the same. One can already suspect that this will come in handy for rewriting charge in termed of something we can actually integrate.

There are really two "regions of interest" in this problem: points inside the sphere, and points outside the sphere. The distribution has spherical symmetry, and so we should be able to "easily" use Gauss's Law to solve for the field.

Gauss's Law works best when you have symmetry of some kind: spherical, planar, linear.

Let's begin by writing down Gauss's Law:

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{A} = q_{enclosed}/\epsilon_0$$

Let's then consider two Gaussian Surfaces: a spherical surface inside the charged sphere, with its own radius r ($r < R$), and a spherical surface that encloses the entire charge sphere and with $r > R$. Just given the symmetry of this sphere, you can already see that whatever the electric field, its vectors will point radially outward from the center of the charge-sphere.

We need to attack that flux integral on the left-hand side. We can use what we have just observed about the charge-sphere to start simplifying things:

- First, given the spherical symmetry of the problem, we already know that $\vec{E}$ is parallel to the normal pointing out from our Gaussian surface piece, $d\vec{A}$. Thus in the dot product, $\vec{E} \cdot d\vec{A} = EdA\cos\theta = EdA$ since $\cos\theta = 1$. So the flux integral just boils down to $\int_{surface} EdA$.
- With our Gaussian surface centered on the charge sphere, it's the same distance from all points along the same radius on the enclosed charge - therefore whatever the field, its value is *constant across the Gaussian surface* and thus can come out of the integral, $\Phi = E\int_{surface} dA$.
- The integral $\int_{surface} dA$ is going to simply yield the total surface area of a sphere of radius r, the radius of our Gaussian surface. Thus $\Phi = EA = E(4\pi r^2)$.

And we've evaluated the surface integral.

Now we need to evaluate the right-hand side of Gauss's Law: $q_{enclosed}/\epsilon_0$. We need to determine the amount of charge enclosed by our Gaussian surface. It will matter whether $r < R$ or $r > R$. For $r > R$, the answer is straight-forward: $q_{enclosed} = Q$, the whole charge of the sphere. What about for $r < R$?

- To tackle this, we need to take advantage of the constant volume density of charge; that is, ρ is the same regardless of the volume inside the sphere we choose to evaluate. Let's consider two different ways of obtaining ρ. We could consider the whole volume of the charge-sphere. In that case,

  $$\rho(R) = \rho = Q/((4/3)\pi R^3).$$

  We could also consider the Gaussian sphere for which $r < R$. In that case, [rho(r) = rho = q_{enclosed}/((4/3) pi r^3).] These two ρs are equivalent, so setting them equal we find that

  $$Q/((4/3)\pi R^3) = q_{enclosed}/(4/3)\pi r^3) \rightarrow q_{enclosed} = Q\left(\frac{r}{R}\right)^3.$$

Let's put all of these pieces together - the right-hand side of Gauss's Law and the left-hand side of Gauss's Law:

- For $r < R$:

  $$\int_{surface} \vec{E} \cdot d\vec{A} = q_{enclosed}/\epsilon_0 \rightarrow E(4\pi r^2) = Q(r/R)^3(1/\epsilon_0).$$

  This reduces to

  $$E(r < R) = \frac{Qr}{4\pi\epsilon_0 R^3}$$

- For $r > R$:

  $$E(4\pi r^2) = Q/\epsilon_0$$

  which reduces to

  $$E(r > R) = \frac{Q}{4\pi\epsilon_0 r^2}.$$

Draw the function on the board, with the linear piece for $r < R$ and the inverse-square law piece for $r > R$.

Does this make sense? Inside the sphere, as we grow the size of our Gaussian surface, we include more and more charge and we expect the field strength to grow. Outside the sphere, we fully enclose all the charge and just get further from the charge as r gets bigger, so we expect this to