Lecture 12

Rate of Convergence

Theorem 5.4

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If A has only r distinct eigenvalues, then the CG iteration will terminate at the solution in at most r iterations.

Proof

Assume eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ take r distinct values:

$$\tau_1 < \tau_2, \ldots, < \tau_r$$

Define polynomial:

$$Q_r(\lambda) = \frac{(-1)^r}{\tau_1 \tau_2 \ldots \tau_r}(\lambda - \tau_1)(\lambda - \tau_2)\ldots(\lambda - \tau_r)$$

$Q_r(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$

$Q_r(0) = 1$

$Q_r(\lambda) - 1$ Is polynomial of degree r with root at

$$\widetilde{P}_{r-1} = \frac{(Q_r(\lambda) - 1)}{\lambda} \qquad \text{Degree } r\text{-1}$$

$$\min_{P_k} \max_{1 \le i \le n} [1 + \lambda_i P_k(\lambda_i)]^2 \quad \text{(B)}$$

$$0 \le \min_{P_{r-1}} \max_{1 \le i \le n} [1 + \lambda_i P_{r-1}(\lambda_i)]^2 \le \max_{1 \le i \le n} [1 + \lambda_i \widetilde{P}_{r-1}(\lambda_i)]^2 = \max_{1 \le i \le n} (Q_r(\lambda_i))^2 = 0$$

$$\min_{P_{r-1}} \max_{1 \le i \le n} [1 + \lambda_i P_{r-1}(\lambda_i)]^2 = 0 \qquad \text{For } k = r\text{-1}$$

From (C)

$$\| x_{k+1} - x^* \|_A^2 \le \min_{P_k} \max_{1 \le i \le n} [1 + \lambda_i P_k(\lambda_i)]^2 \, \| x_0 - x^* \|_A^2 = 0$$

$$\| x_r - x^* \|_A^2 = 0$$

Therefore

$$x_r = x^*$$

QED

Theorem 5.5

If A has eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ we have

$$\| x_{k+1} - x^* \|_A^2 \le \left(\frac{\lambda_{n-k} - \lambda_1}{\lambda_{n-k} + \lambda_1} \right)^2 \| x_0 - x^* \|_A^2$$

Eigenvalues

$\lambda_1, \ldots, \lambda_{n-k}, \lambda_{n-k+1}, \ldots, \lambda_n$

Eigenvalues

$\lambda_1, \ldots, \lambda_{n-k}, \lambda_{n-k+1}, \ldots, \lambda_n$

Select polynomial of degree k such that

Q has roots at k largest eigenvalues

$\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-k+1}$

As well as at mid point λ_1 and λ_{n-k}

$$Q_{k+1}(\lambda) = 1 + \lambda \overline{P}_k(\lambda)$$

Maximum value attained by Q on the remaining eigenvalues is precisely

$$\left(\frac{\lambda_{n-k} - \lambda_1}{\lambda_{n-k} + \lambda_1} \right)$$

(C) $\| x_{k+1} - x^* \|_A^2 \le \min_{P_k} \max_{1 \le i \le n} [1 + \lambda_i P_k(\lambda_i)]^2 \| x_0 - x^* \|_A^2$

$$\| x_{k+1} - x^* \|_A^2 \le \left(\frac{\lambda_{n-k} - \lambda_1}{\lambda_{n-k} + \lambda_1} \right)^2 \| x_0 - x^* \|_A^2$$