6.003: Signals and Systems

Convolution

October 15, 2009

Multiple Representations of CT and DT Systems

Verbal descriptions: preserve the rationale.

Difference/differential equations: mathematically compact.

$$y[n] = x[n] + z_0 y[n-1] \qquad \dot{y}(t) = x(t) + s_0 y(t)$$

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.

$$\frac{Y}{X} = \frac{1}{1 - z_0\mathcal{R}} \qquad \frac{Y}{X} = \frac{\mathcal{A}}{1 - s_0\mathcal{A}}$$

Transforms: representing diff. equations with algebraic equations.

$$H(z) = \frac{z}{z - z_0} \qquad H(s) = \frac{1}{s - s_0}$$

Convolution

Representing a system by a single signal.

Responses to arbitrary signals

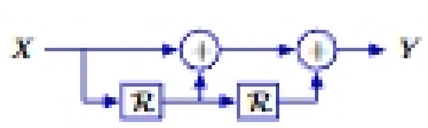
Although we have focused on responses to simple signals ($\delta[n]$, $\delta(t)$) we are generally interested in responses to more complicated signals.

How do we compute responses to a more complicated input signals?

No problem for difference equations / block diagrams.
→ use step-by-step analysis.

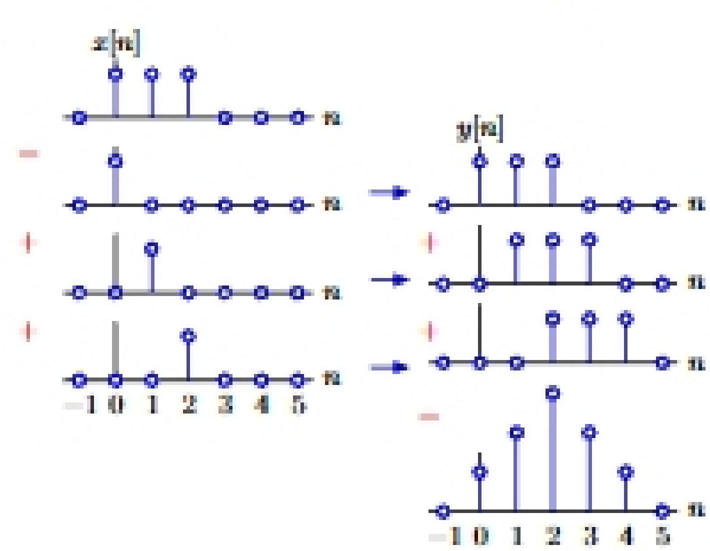
Check Yourself

What is $y[3]$?



Superposition

Break input into additive parts and sum the responses to the parts.



Superposition works if the system is **linear**.

Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

$$x_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n]$$

and

$$x_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n]$$

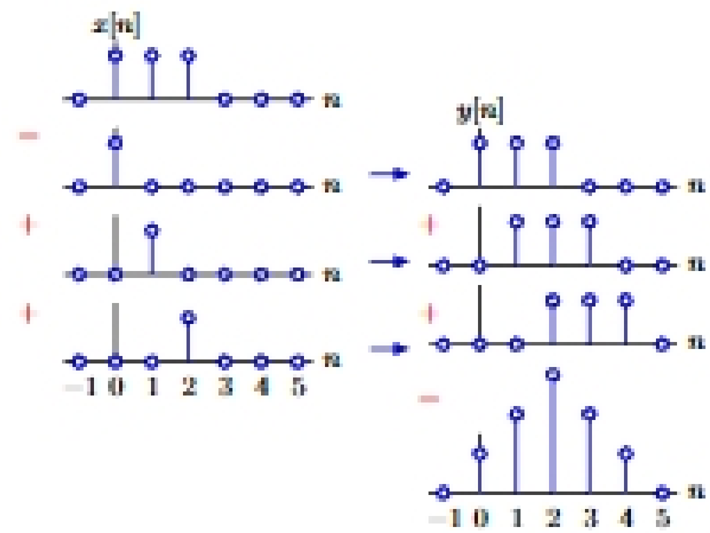
the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \boxed{\text{system}} \rightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β.

Superposition

Break input into additive parts and sum the responses to the parts.



Reponses to parts are easy to compute if system is **time-invariant**.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

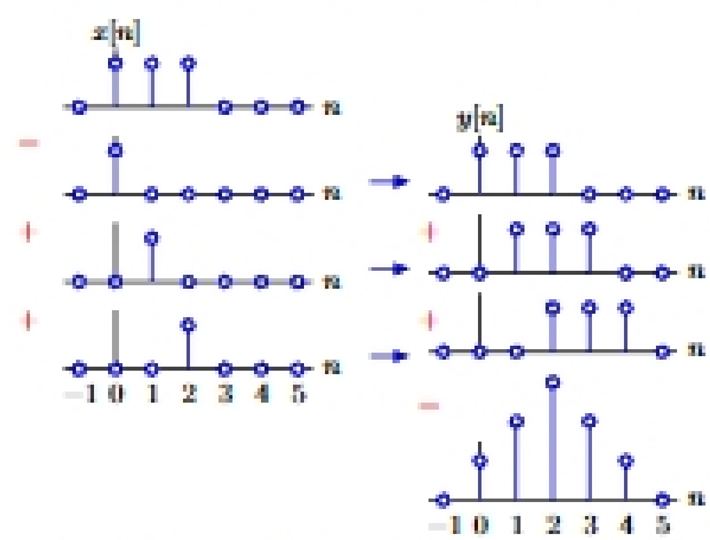
the system is time invariant if

$$x[n-n_0] \rightarrow \boxed{\text{system}} \rightarrow y[n-n_0]$$

is true for all n_0.

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition is easy if the system is **linear** and **time-invariant**.

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \rightarrow \boxed{\text{system}} \rightarrow h[n]$$

$$\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow h[n-k]$$

$$x[k]\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution

Response of an LTI system to an arbitrary input.

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called **convolution**.

Notation

Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = (x*h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x*h)[n] = x[n]*h[n]$$

Notation

Do not be fooled by the confusing notation.

Confusing (but conventional) notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$x[n]*h[n]$ looks like an operation of samples; but it is not!

$$x[1]*h[1] \neq (x*h)[1]$$

Convolution operates on signals not samples.

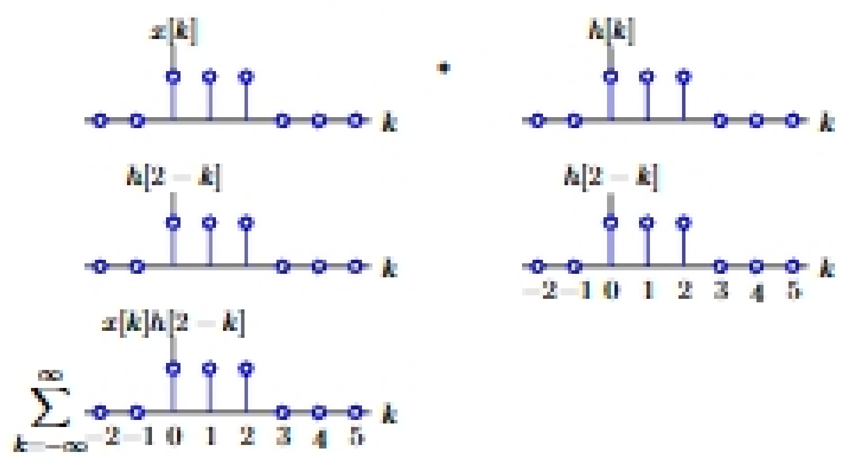
Unambiguous notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = (x*h)[n]$$

The symbols x and h represent DT signals.
Convolving x with h generates a new DT signal $x*h$.

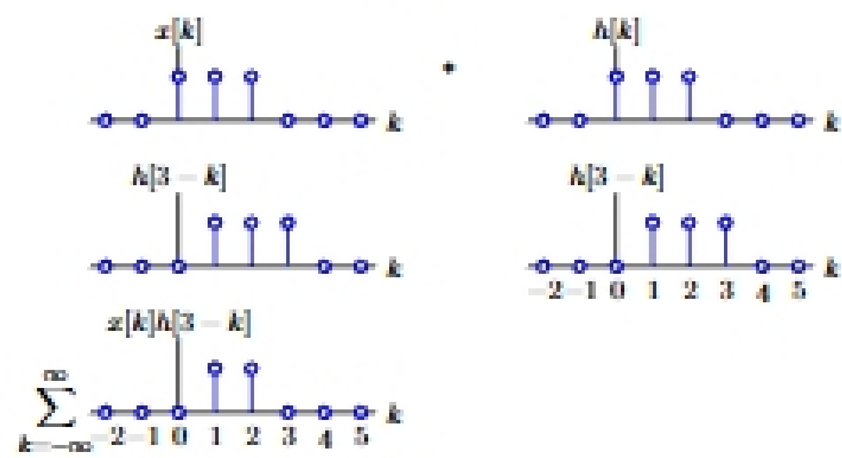
Structure of Convolution

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



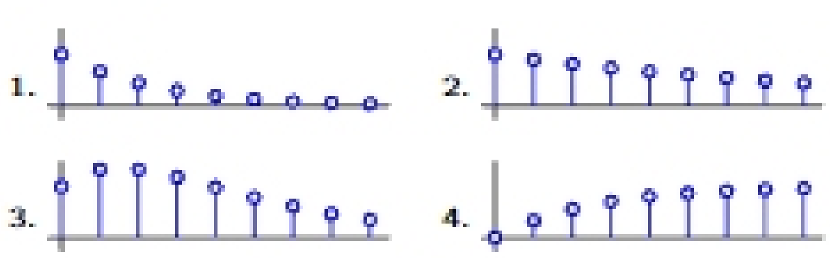
Structure of Convolution

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



Check Yourself

Which plot shows the result of the convolution above?



5. none of the above

Convolution

Representing an LTI system by a single signal.

Unit-sample response $h[n]$ is a complete description of an LTI system.

Given $h[n]$ one can compute the response $y[n]$ to any arbitrary input signal $x[n]$:

$$y[n] = (x*h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$