Section C.1: The Exponential Function

Exponential function

A *exponential function* grows or decay by the same relative amount per unit time. For any quantity Q growing exponentially with a fractional growth rate r

$$Q = Q_0 \times (1 + r)^t$$

where

Q = value of the exponentially growing quantity at time t

Q_0 = initial value of the quantity (at $t = 0$)

r = fractional growth rate (which may be positive or negative) for the quantity

t = time

Negative values of r correspond to exponential decay. Note that the units of time used for t and r must be the same. For example, if the fractional growth rate is 0.05 per month, then t must also be measured in months.

Remark

You may notice that the exponential function is identical to the compound interest formula if we identify Q as the accumulate balance A, Q_0 as the starting principal P, r as the interest rate, and t as the number of times interest is paid. In other words, compound interest is a form of exponential growth.

Ex.1 U.S. population growth.

The 2000 census found a U.S. population of about 281 million with an estimated growth rate of 0.7% per year. Write an equation for the U.S. population that assumes exponential growth at this rate. Use the equation to predict the U.S. population in 2100.

Ex.2 Declining population.
China's one-child policy was originally implemented with the goal of reducing China's population to 700 million by 2050. China's population was about 1.2 billion. Suppose China's 2000 population declines at a rate of 0.5% per year. Write an equation for the exponential decay of the population. Will this rate of decline be sufficient to meet the original goal?

Graphing Exponential Functions

How to graph exponential functions

The easiest way to graph an exponential function is to use points corresponding to several doubling times (or half-lives in the case of decay), that is the points

$$(0, Q_0),\ (T_{double}, 2Q_0),\ (2T_{double}, 4Q_0)\cdots$$

in the case of exponential growth, or

$$(0, Q_0),\ (T_{half}, \frac{Q_0}{2}),\ (2T_{half}, \frac{Q_0}{4})\cdots$$

in the case of exponential decay.

Ex.3

The growth rate of the U.S. population has varied substantially during the past century. It depends on the immigration rate, as well as birth and death rates. Starting from a 2000 population of 281 million, project the population in 2100 using growth rates that are just 0.2 percentage point lower and higher than the 0.7% used in Example 1. Make a graph showing the population through 2100 for each growth rate.