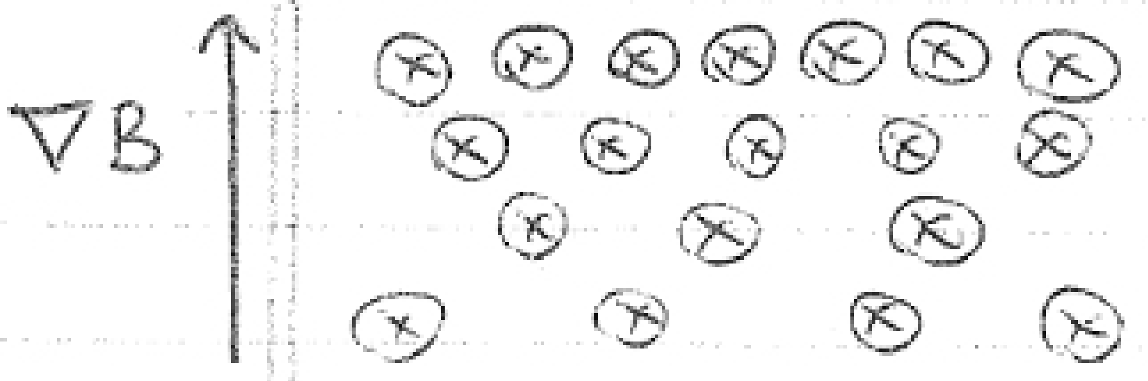
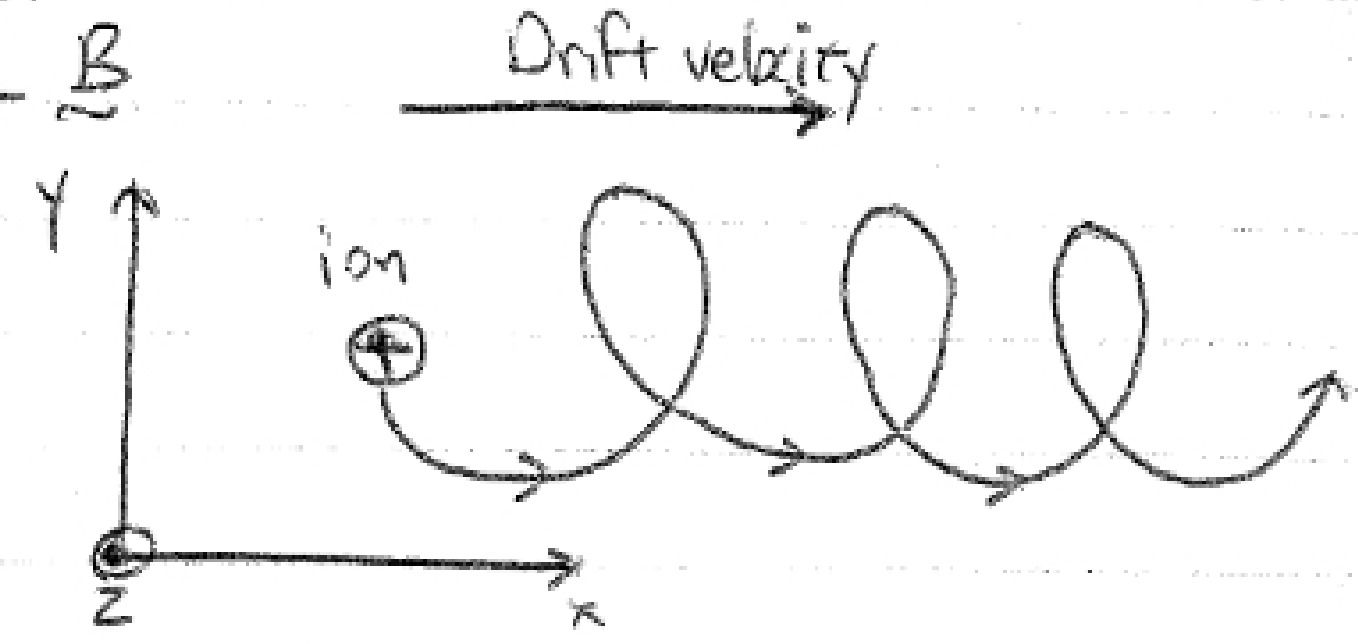
Lecture #5: Particle Motion in Constant, Non-uniform $\underset{\sim}{B}$ Fields   Howes SD

I. The "Grad B" Drift:

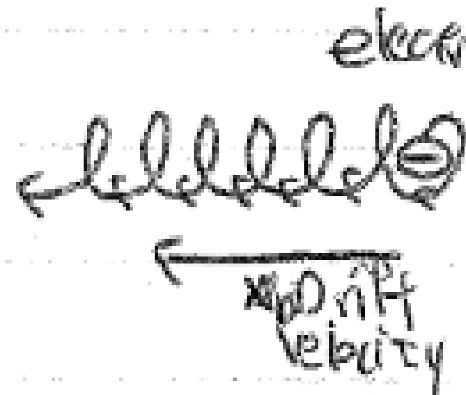
A. Simplest Case: $\nabla|\underset{\sim}{B}| \perp \underset{\sim}{B}$

1.





2. Ions and electrons drift in opposite directions!



3. Two Spatial scales characteristic of this problem:
   a. Larmor radius $r_L = \frac{v_\perp}{\omega_c}$
   b. Magnetic Field Scale Length $L = \left(\frac{\nabla B}{B}\right)^{-1}$

4. This problem can be done using a multiple-scale analysis using an expansion parameter

$$\epsilon = \frac{r_L}{L} \ll 1$$

   a. In this case, $\underset{\sim}{B}$ changes little over the Larmor radius.
   b. In general, for the case of $\underset{\sim}{E} = 0$,

$$m\frac{d\underset{\sim}{v}}{dt} = q\, \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}) \leftarrow$$ We can expand in ϵ, solving order by order.

~~[illegible]~~

   c. But, the solution in general requires a lot of algebra and averaging, so rather than that we'll take a more "intuitive" approach.

Lecture #5 (Continued) Howes ③

I. (Continued)

B. Intuitive approach based on average force.

1. Analogous to $\underset{\sim}{E} \times \underset{\sim}{B}$ or Gravitational drift, the drift due to a general force $\underset{\sim}{F}$ (for $m\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{F}$)

is $\boxed{\underset{\sim}{V}_D = \frac{1}{q}\frac{\underset{\sim}{F} \times \underset{\sim}{B}}{B_0^2}}$

2. We'll use a perturbative approach to find the <u>effective averaged force</u> due to a gradient in B, then use the formula above to find the drift.

3. $$\boxed{\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt} = q\left[\underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r})\right]}$$

4. Taylor Expansion: $\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}(\underset{\sim}{r}_0) + (\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\nabla\underset{\sim}{B} + \frac{\left[(\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\nabla\right]^2}{2!}\underset{\sim}{B} + \ldots$

a. Note, if we normalize this formula according to $B_0 = |\underset{\sim}{B}(\underset{\sim}{r}_0)|$, we find

$$\frac{\underset{\sim}{B}(\underset{\sim}{r})}{B_0} = \underbrace{\hat{\underset{\sim}{b}}}_{\mathcal{O}(1)} + \underbrace{(\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\frac{\nabla\underset{\sim}{B}}{B_0}}_{\mathcal{O}\left(r_L \cdot \frac{(B_0/L)}{B_0}\right) = \mathcal{O}\left(\frac{r_L}{L}\right)}$$

b. Thus, as usual, our Taylor Expansion is a good approximation if

$$\epsilon = \frac{r_L}{L} \ll 1.$$

5. a. To simplify the algebra, consider $\underset{\sim}{B}(y) = B_0\hat{\underset{\sim}{z}} + \epsilon\, y\frac{\partial B}{\partial y}\hat{\underset{\sim}{z}} \ldots$

b. Expand $\underset{\sim}{V} = \underbrace{\underset{\sim}{v}_0}_{\substack{\text{unperturbed}\\\text{orbit for}\\\nabla B = 0}} + \epsilon\underbrace{\underset{\sim}{v}_1}_{\substack{\text{correction to}\\\text{orbit due to}\\\nabla B}} + \ldots$

6. From Lecture #3 for $\underset{\sim}{E}=0$, we know the orbit in the absence of the gradient gives

(take $\hat{\underset{\sim}{e}}_1 = \hat{\underset{\sim}{x}}$ & $\hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{y}}$)

$$\left.\begin{aligned} V_x &= V_\perp \cos\omega_c t \\ V_y &= -V_\perp \sin\omega_c t \\ V_z &= V_{\parallel 0} \end{aligned}\right\} \underset{\sim}{V} = V_\perp(\cos\omega_c t\,\hat{\underset{\sim}{x}} - \sin\omega_c t\,\hat{\underset{\sim}{y}}) + V_{\parallel 0}\hat{\underset{\sim}{z}}$$

I. B. (Continued)

7. $\underset{\sim}{F} = q(\underset{\sim}{v}_0 + \epsilon \underset{\sim}{v}_1) \times \left(B_0 \hat{\underset{\sim}{z}} + \epsilon y \frac{\partial B}{\partial y} \hat{\underset{\sim}{z}}\right)$

a. $\mathcal{O}(1)$: $\underset{\sim}{F}_0 = q\left(\underset{\sim}{v}_0 \times B_0 \hat{\underset{\sim}{z}}\right) = qB_0 v_\perp \left(-\cos\omega_c t\, \hat{\underset{\sim}{y}} - \sin\omega_c t\, \hat{\underset{\sim}{x}}\right)$

To find average force, take $\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \underset{\sim}{F}_0\, dt = \langle \underset{\sim}{F}_0 \rangle$

$$\langle \underset{\sim}{F}_0 \rangle = \frac{qB_0 v_\perp \omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \left(\cancelto{0}{-\cos\omega_c t\, \hat{\underset{\sim}{y}}} - \cancelto{0}{\sin\omega_c t\, \hat{\underset{\sim}{x}}}\right) dt = 0$$

b. $\mathcal{O}(\epsilon)$: $\underset{\sim}{F}_1 = q \underbrace{\underset{\sim}{v}_1 \times (B_0 \hat{\underset{\sim}{z}})}_{①} + q \underset{\sim}{v}_0 \times \underbrace{\left(y \frac{\partial B_y}{\partial y} \hat{\underset{\sim}{z}}\right)}_{②}$

① We annihilate term ① by assuming $\underset{\sim}{v}_1$ is periodic in $T = \frac{2\pi}{\omega_c}$.
To verify this assumption requires substantial algebra, skipped here.

$$\frac{qB_0\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \underset{\sim}{v}_1 \times \hat{\underset{\sim}{z}} = 0$$

② From our solution to $\underset{\sim}{v}_0$, we get $y - y_0 = \int \frac{dv_y}{dt} dt = \frac{v_\perp}{\omega_c}\cos\omega_c t$
we'll take $y_0 = 0$ for simplicity.

$$\left\langle q \underset{\sim}{v}_0 \times \left(y \frac{\partial B_y}{\partial y} \hat{\underset{\sim}{z}}\right)\right\rangle = \frac{q\omega_c}{2\pi}\frac{\partial B_y}{\partial y}\frac{v_\perp^2}{\omega_c}\int_0^{\frac{2\pi}{\omega_c}} \cos\omega_c t \left(\cos\omega_c t\, \hat{\underset{\sim}{y}} - \sin\omega_c t\, \hat{\underset{\sim}{x}}\right) dt$$

NOTE: $\int_0^{\frac{2\pi}{a}} \cos^2 ax\, dx = \frac{\pi}{a}$, $\int_0^{2\pi} \cos x \sin x\, dx = 0$

$$\boxed{\langle \underset{\sim}{F}_1 \rangle = \frac{-q v_\perp^2}{2\omega_c}\frac{\partial B_y}{\partial y}\hat{\underset{\sim}{y}}}$$

Thus, there is an average force in the direction of the gradient.

8. ~~[illegible]~~

For a general gradient $\nabla B \perp \underset{\sim}{B}$, $\langle \underset{\sim}{F}_1 \rangle = \frac{-q v_\perp^2}{2\omega_c}\nabla B$