5.3 Determinants and Cramer's Rule

Unique Solution of a 2×2 System

The 2×2 system

$$\begin{array}{rcrcl} ax & + & by & = & e, \\ cx & + & dy & = & f, \end{array} \tag{1}$$

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

$$x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}. \tag{2}$$

This result, called **Cramer's Rule** for 2×2 systems, is usually learned in college algebra as part of determinant theory.

Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Evaluation of a 2×2 determinant is by **Sarrus' Rule**:

$$\begin{vmatrix} \mathbf{a} & b \\ c & \mathbf{d} \end{vmatrix} = \mathbf{ad} - bc.$$

The boldface product **ad** is the product of the main diagonal entries and the other product bc is from the anti-diagonal.

Cramer's 2×2 rule in determinant notation is

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{3}$$

Unique Solution of an $n \times n$ System

Cramer's rule can be generalized to an $n \times n$ system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ or

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n. \end{array} \tag{4}$$

System (4) has a unique solution provided the **determinant of coefficients** $\Delta = \det(A)$ is nonzero, in which case the solution is given by

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ \ldots,\ x_n = \frac{\Delta_n}{\Delta}. \tag{5}$$

The determinant Δ_j equals $\det(B_j)$ where matrix B_j is matrix A with column j replaced by $\vec{\mathbf{b}} = (b_1, \ldots, b_n)$, which is the right side of system (4). The result is called **Cramer's Rule** for $n \times n$ systems. Determinants will be defined shortly; intuition from the 2×2 case and Sarrus' rule should suffice for the moment.

Determinant Notation for Cramer's Rule. The **determinant of coefficients** for system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is denoted by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{6}$$

The other n determinants in Cramer's rule (5) are given by

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}. \tag{7}$$

The literature is filled with conflicting notations for matrices, vectors and determinants. The reader should take care to use vertical bars *only* for determinants and absolute values, e.g., $|A|$ makes sense for a matrix A or a constant A. For clarity, the notation $\det(A)$ is preferred, when A is a matrix. The notation $|A|$ implies that *a determinant is a number*, computed by $|A| = \pm A$ when $n = 1$, and $|A| = a_{11}a_{22} - a_{12}a_{21}$ when $n = 2$. For $n \geq 3$, $|A|$ is computed by similar but increasingly complicated formulas; see Sarrus' rule and the *four properties* below.

Sarrus' Rule for 3×3 Matrices. College algebra supplies the following formula for the determinant of a 3×3 matrix A:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad -a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned} \tag{8}$$

The number $\det(A)$ can be computed by an algorithm similar to the one for 2×2 matrices, as in Figure 10. We remark that no further generalizations are possible: *there is no Sarrus' rule for* 4×4 *or larger matrices*!

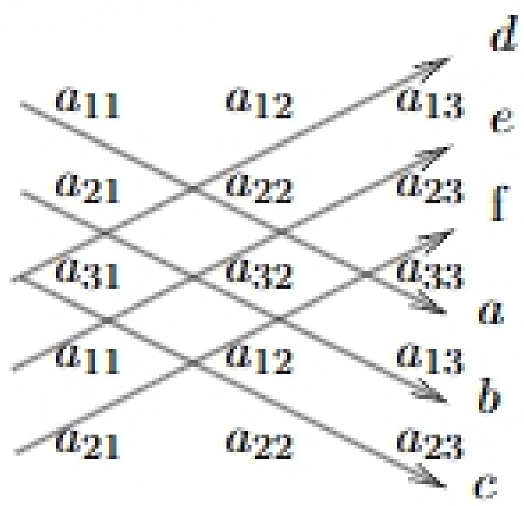

Figure 10. Sarrus' rule for 3×3 matrices, which gives

$$\det(A) = (a + b + c) - (d + e + f).$$

College Algebra Definition of Determinant. The impractical definition is the formula

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}. \tag{9}$$

In the formula, a_{ij} denotes the element in row i and column j of the matrix A. The symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ stands for a rearrangement of the subscripts 1, 2, ..., n and S_n is the set of all possible rearrangements. The nonnegative integer parity(σ) is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers $\sigma_1, \ldots, \sigma_n$ into natural order 1, ..., n.

Formula (9) reproduces the definition for 3×3 matrices given in equation (8). We will have no computational use for (9). For computing the value of a determinant, see below *four properties* and *cofactor expansion*.

Four Properties. The definition of determinant (9) implies the following four properties:

Triangular	The value of $\det(A)$ for either an upper triangular or a lower triangular matrix A is the product of the diagonal elements: $\det(A) = a_{11}a_{22} \cdots a_{nn}$.
Swap	If B results from A by swapping two rows, then $\det(A) = (-1)\det(B)$.
Combination	The value of $\det(A)$ is unchanged by adding a multiple of a row to a different row.
Multiply	If one row of A is multiplied by constant c to create matrix B, then $\det(B) = c\det(A)$.

It is known that these four rules suffice to compute the value of any $n \times n$ determinant. The proof of the four properties is delayed until page 320.