

# ME 201/MTH281/ME400/CHE400

## Laplace Equation in a Cylinder - Oscillatory Behavior in z.

In this notebook, we construct contour plots of a solution of the Laplace equation obtained in class. The problem solved was to find the potential in a circular cylinder with zero potential on the bottom and top, and with a specified potential (a quadratic function of  $z$  in this case) on the side. The mathematical formulation of the problem is given below.

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad r < a \text{ and } 0 < z < h,$$

$$\text{with } \Phi(r, 0) = 0, \quad \Phi(r, h) = 0, \text{ and } \Phi(a, z) = \Phi_0 (z/h) (1 - z/h),$$

where  $\Phi_0 = \text{constant}$ . We solved this in class by separation of variables. The solution is

$$\Phi(r, z) = \frac{8 \Phi_0}{\pi^3} \sum_{n=1, n \text{ odd}}^{\infty} \frac{I_0(n\pi r/h)}{I_0(n\pi a/h)} \frac{\sin(n\pi z/h)}{n^3}.$$

where  $I_0$  is the modified Bessel function of the first kind of order zero. We will use this solution as the basis for constructing contour plots of the potential in the cylinder. We begin by specifying the parameter values.

**Besseli**

```
in(1)- h = 4.0 (** m **); a = 2.0 (** m **); Phi0 = 100.0 (** volts **);
```

Now we define the  $n$ th term in the series.

```
in(2)- term[r_, z_, n_] :=
  ((8 Phi0) / pi^3) (Besseli[0, (n pi r) / h] / Besseli[0, (n pi a) / h]) (Sin[(n pi z) / h] / n^3)
```

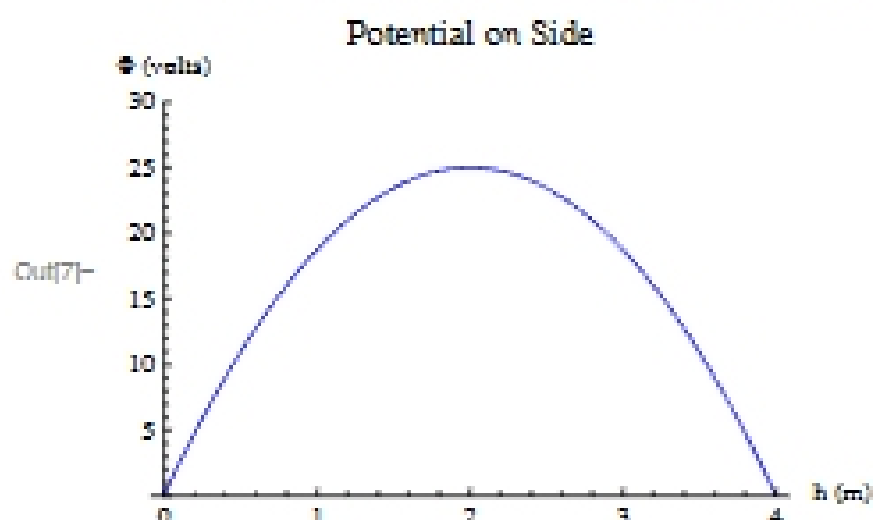
Finally we define the  $k$ th partial sum.

```
in(3)- sol[r_, z_, k_] := Sum[term[r, z, n], {n, 1, k, 2}]
```

We begin by checking the boundary function on the top of the cylinder. We set the option `ImageSize` to 250 for all of our plot statements. Because the boundary function satisfies the same boundary conditions as the  $z$ -eigenfunctions, we expect rapid convergence. We use 21 terms in the series. We plot the exact function in red, the series sum in blue.

```
in(4)- SetOptions[{Plot, ContourPlot}, ImageSize -> 250];
```

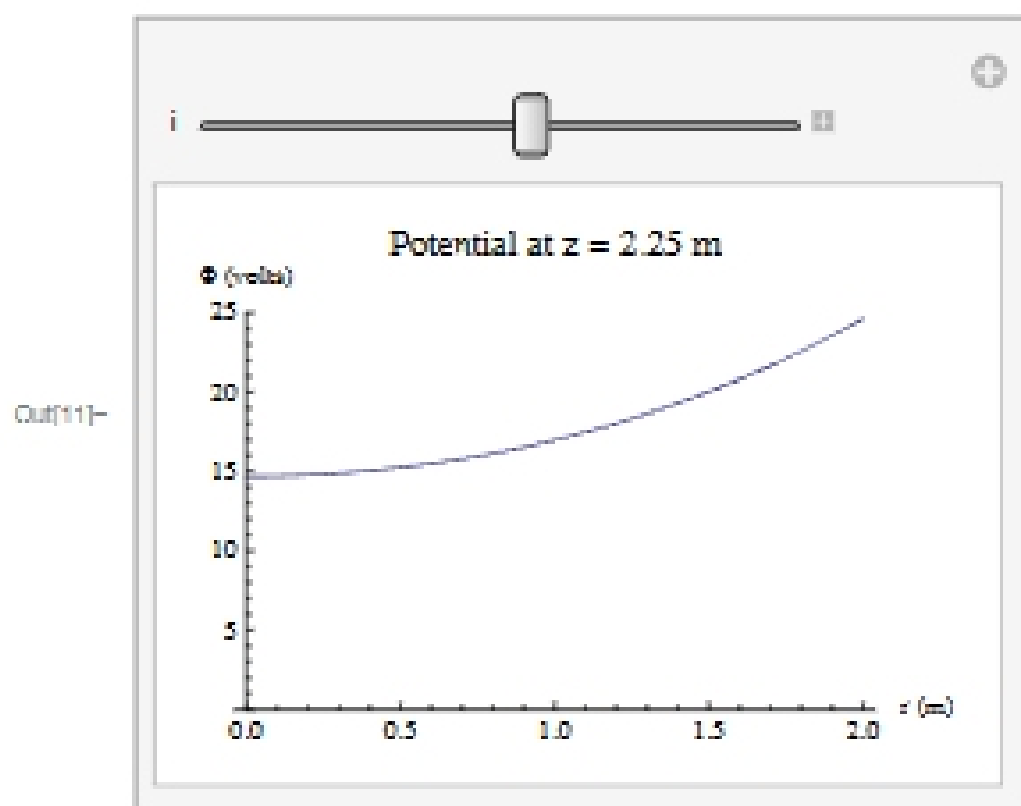
```
in(7)- Plot[{sol[a, z, 21], Phi0 (z/h) (1 - z/h)}, {z, 0, h}, AxesLabel -> {"h (m)", "Phi (volts)"},
  PlotLabel -> "Potential on Side", PlotRange -> {0, 30}, PlotStyle -> {Red, Blue}]
```



The exact function and the series representation agree.

Now we use `Manipulate` to display a graph sequence, showing the profiles of potential versus radius for many values of  $z$ . We first define a function `profile[z]` which returns the graph for the given value of  $z$ . We write the routine so that 20 terms are used in the series.

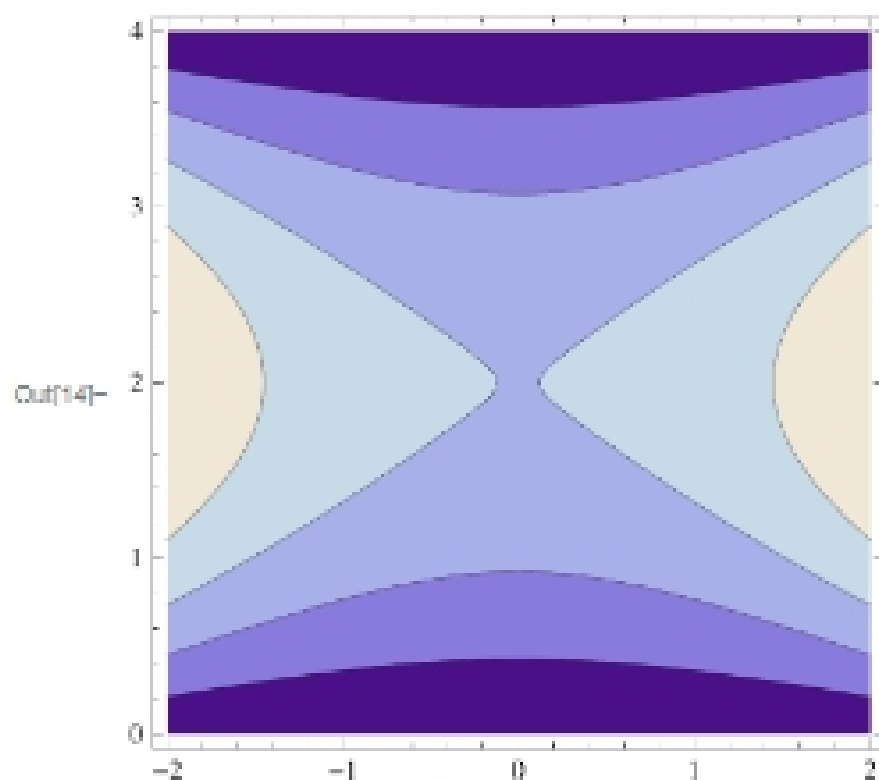
```
In[10]- profile[z_] := Module[{terms}, terms = 20;
  Plot[sol[r, z, terms], {r, 0, a}, AxesLabel -> {"r (m)", "Φ (volts)"}, PlotLabel ->
    Row[{"Potential at z =", PaddedForm[z, {3, 2}], " m"}], PlotRange -> {0, 25}]
In[11]- DynamicModule[{mangraph, i}, Do[mangraph[i] = profile[(h + i) / 80.], {i, 0, 80}];
  Manipulate[mangraph[i], {i, 0, 80, 1}]]
```



By using the slider, we can see the radial profile of potential at any height in cylinder (more accurately, at any of 81 given heights in the cylinder). The movie plays smoothly and shows how the potential drops off with height most rapidly near the top of the cylinder.

As a final graph, we construct contours of constant potential in the cylinder. We use 20 terms in the partial sums. To keep the geometry true, we use an aspect ratio equal to the height to width ratio. We ask for contours at 5 volt intervals from 5 to 20. The upper and lower boundaries are the zero volt contours.

```
In[14]- ContourPlot[sol[r, z, 20], {r, -a, a}, {z, 0, h}, PlotPoints -> 100,
  AspectRatio -> h / (2 a), Contours -> {5, 10, 15, 20}, ContourShading -> True]
```



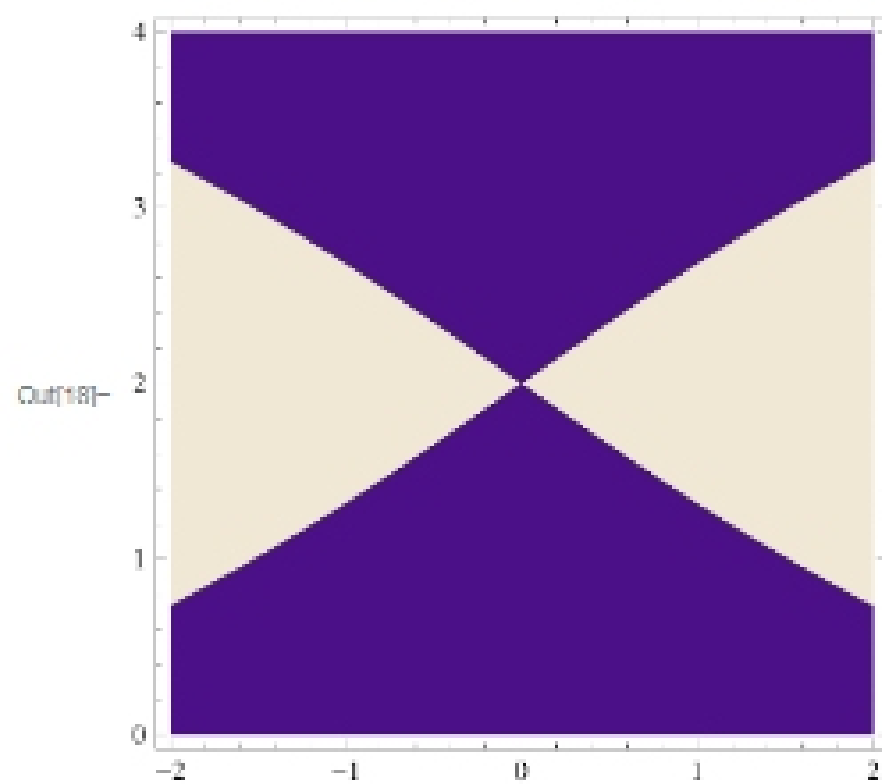
If you use the mouse to place the cursor on any curve, *Mathematica* will return the voltage of that contour.

We check the potential in the center of the cylinder.

```
In[15]- sol[0, h / 2, 21]
```

```
Out[15]- 14.9668
```

```
In[18]- ContourPlot[sol[r, z, 20], {r, -a, a}, {z, 0, h}, PlotPoints -> 100,
  AspectRatio -> h / (2 a), Contours -> {14.9668}, ContourShading -> True]
```



The potential in the purple regions is greater than 14.9668, and the potential in the tan regions is less than 14.9668.

We can add a few more contours.

```
In[19]- ContourPlot[sol[r, z, 20], {r, -a, a}, {z, 0, h}, PlotPoints -> 100,
  AspectRatio -> h / (2 a), Contours -> {5, 10, 14.9668}, ContourShading -> True]
```

