Calculating DFT

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N}=\sum_{n=0}^{N-1}x(n)\left(e^{-j2\pi/N}\right)^{nk}$$

Denote $W_N=e^{-j2\pi/N}$, then $X(k)=\sum_{n=0}^{N-1}x(n)W_N^{nk}$

- Each X(k) requires N complex multiplications and N complex additions.
- Each complex multiplication needs 4 real multiplications and 2 real addition because (a+jb)*(c+jd) = (ac-bd)+j(bc+ad)
- There are N different X(k), so we need a total of

  $4N^2$ real multiplications

  $4N^2$ real additions

However, by taking advantages of some properties of W_N^m, we can significantly speed out the calculations ⇒ Fast Fourier Transform

Interesting Properties of W_N^m:

(1) $W_N^0=(e^{-j2\pi/N})^0=e^0=1,\qquad W_N^N=e^{-j2\pi}=1$

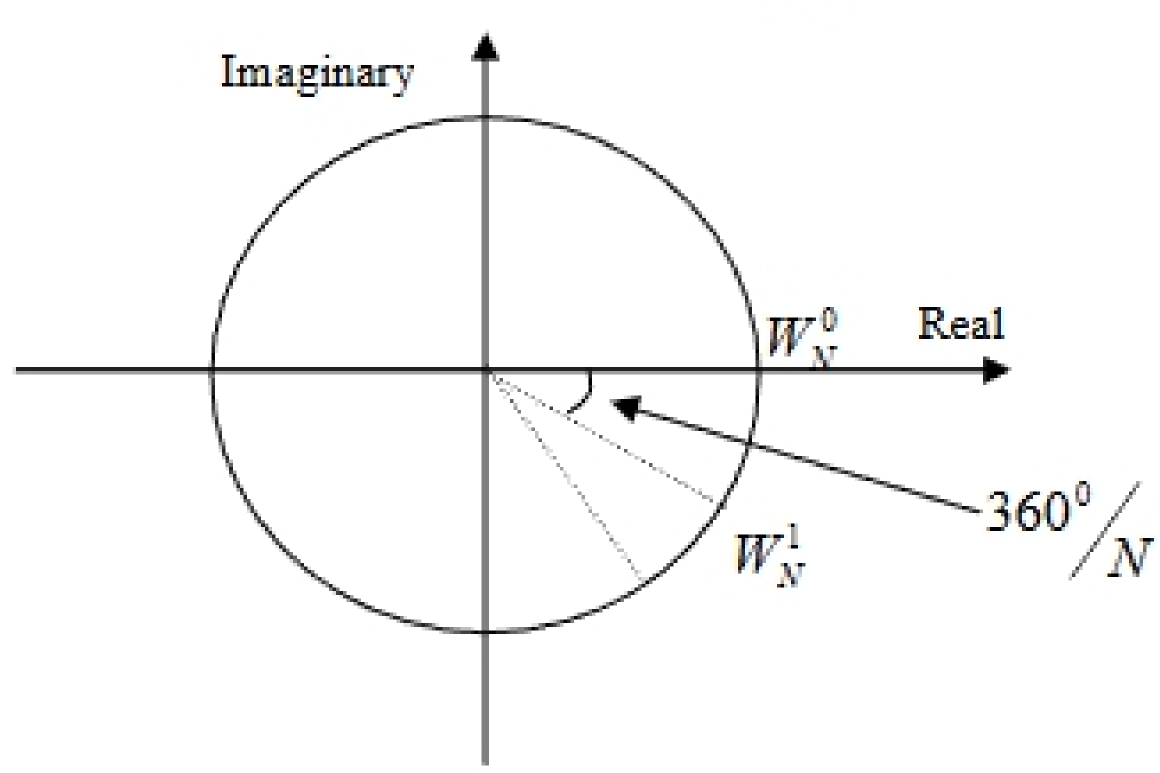
(2) $W_N^{N+m}=W_N^m$ (periodic)

(3) Assume N is a multiple of 4

$$W_N^{N/2}=e^{-j2\pi(N/2)/N}=e^{-j\pi}=-1$$
$$W_N^{N/4}=e^{-j2\pi(N/4)/N}=e^{-j\pi/2}=-j$$
$$W_N^{3N/4}=e^{-j2\pi(3N/4)/N}=e^{-j3\pi/2}=j$$

(4) Assume N is even

$$W_N^{2kr}=(e^{-j2\pi/N})^{2kr}=(e^{-j2\pi/(N/2)})^{kr}=W_{\frac{N}{2}}^{kr}$$

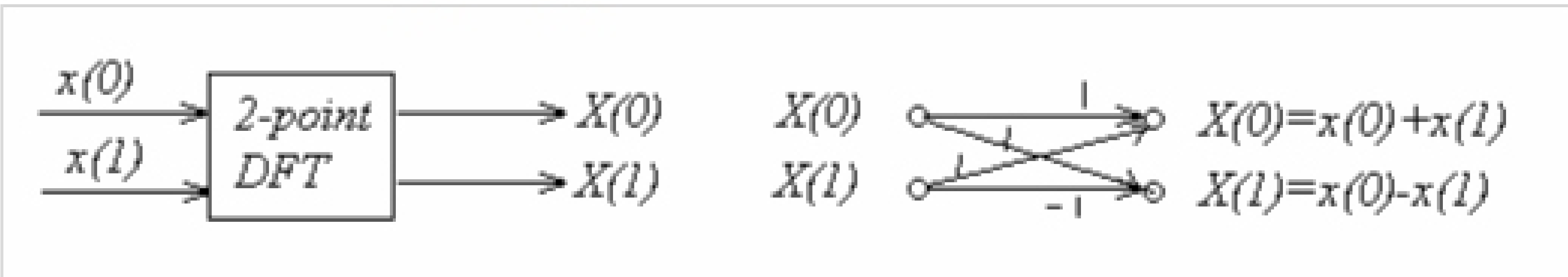


Example 10-3: Two-Point DFT

$$x(0), x(1): \quad X(k) = \sum_{n=0}^{1} x(n) W_2^{nk} \qquad k = 0,1$$

$$X(0) = \sum_{n=0}^{1} x(n) W_2^{n0} = \sum_{n=0}^{1} x(n) = x(0) + x(1)$$

$$\begin{aligned} X(1) &= \sum_{n=0}^{1} x(n) W_2^{n1} = \sum_{n=0}^{1} x(n) W_2^{n} \\ &= x(0) W_2^{0} + x(1) W_2^{1} \\ &= x(0) + x(1)(-1) \\ &= x(0) - x(1) \end{aligned}$$



Example 10-4: Four-point DFT of $x(0)$, $x(1)$, $x(2)$, $x(3)$

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk} \qquad k = 0,1,2,3,$$

$$X(0) = \sum_{n=0}^{3} x(n) W_4^{n0} = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$\begin{aligned} X(1) &= \sum_{n=0}^{3} x(n) W_4^{n} = x(0) W_4^{0} + x(1) W_4^{1} + x(2) W_4^{2} + x(3) W_4^{3} \\ &= x(0) - jx(1) - x(2) + jx(3) \end{aligned}$$

$$\begin{aligned} X(2) &= \sum_{n=0}^{3} x(n) W_4^{2n} = x(0) W_4^{0} + x(1) W_4^{2} + x(2) W_4^{4} + x(3) W_4^{6} \\ &= x(0) + x(1)(-1) + x(2)(1) + x(3) W_4^{2} \\ &= x(0) - x(1) + x(2) - x(3) \end{aligned}$$

$$X(3) = \sum_{n=0}^{3} x(n)W_4^{3n} = x(0)W_4^{0} + x(1)W_4^{3} + x(2)W_4^{6} + x(3)W_4^{9}$$
$$= x(0) + x(1)W_4^{3} + x(2)(1)W_4^{2} + x(3)W_4^{1}$$
$$= x(0) + jx(1) + (-1)x(2) + (-j)x(3)$$
$$= x(0) + jx(1) - x(2) - jx(3)$$

$$\Rightarrow \begin{aligned} X(0) &= [x(0)+x(2)] + [x(1)+x(3)] \\ X(1) &= [x(0)-x(2)] + (-j)[x(1)-x(3)] \\ X(2) &= [x(0)+x(2)] - [x(1)+x(3)] \\ X(3) &= [x(0)-x(2)] + j[x(1)-x(3)] \end{aligned}$$

If we denote $z(0) = x(0)$, $z(1) = x(2)$ => $Z(0) = z(0) + z(1) = x(0) + x(2)$
$Z(1) = z(0) - z(1) = x(0) - x(2)$

Two-point DFT of [x(0) x(2)]

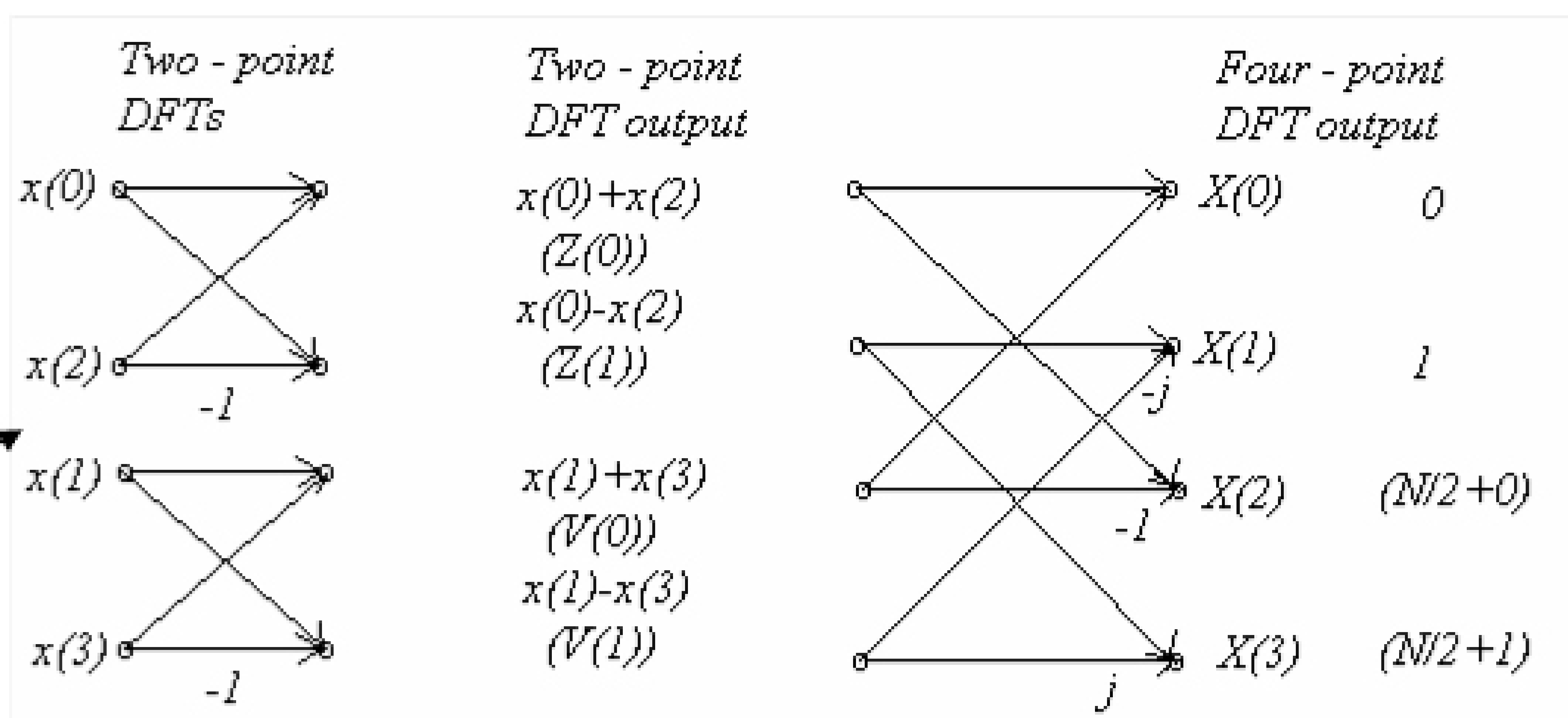
$v(0) = x(1)$, $v(1) = x(3)$ => $V(0) = v(0) + v(1) = x(1) + x(3)$
$V(1) = v(0) - v(1) = x(1) - x(3)$

Two-point DFT of [x(1) x(3)]

Our four – point DFT:

$X(0) = Z(0) + V(0)$
$X(1) = Z(1) + (-j)V(1)$
$X(2) = Z(0) - V(0)$
$X(3) = Z(1) + jV(1)$

Structure like this is called "butterfly"



Key point: We compute 4-point DFT based on two 2-point DFTs