Differential Equations and Linear Algebra

2250-4 at 10:15am on 17 Dec 2010

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on **WEBCT**. Recording errors are reported by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90+91+92+89+89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

Name _____ **12:25pm**

Mathematics 2250-4 Final Exam at 10:15am on 17 Dec 2010

Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.
Part 1. [5%]: ☐ True or ☐ False:
If the 3×3 matrices A and B are upper triangular, then the product AB is triangular.
Part 2. [5%]: ☐ True or ☐ False:
If a 3×3 matrix A has determinant zero, then for all vectors $\mathbf{b}$,
the equation $A\mathbf{x} = \mathbf{b}$ has infinitely solutions $\mathbf{x}$.

Answer: True, False.

[40%] **Ch3(b)**: Determine which values of k correspond to infinitely many solutions for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer: There is a unique solution for $\det(A) \neq 0$, which is equivalent to $k \neq 2$ and $k \neq 3$. Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$. Then (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 2$.

[30%] **Ch3(c)**: Define matrix A and vector $\mathbf{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\mathbf{x} = \mathbf{b}$.

Answer: $x_2 = \Delta_2/\Delta$, $\Delta_2 = \det \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} = 20$, $\Delta = \det(A) = 4$, $x_3 = 5$.

[20%] **Ch3(d)**: Assume the transpose of A is $A^T = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the transpose of the inverse of A.

Answer: Then $(A^{-1})^T = (A^T)^{-1} = \left(\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix} \right)^{-1} = \frac{1}{8} \begin{pmatrix} 4 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$.

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Name _____ **12:25pm**

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove that 1, x, $x\sin(x)$ are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Rank test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Wronskian test**	The Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sample test**	Define $\mathbf{v}(x) = \big(f_1(x), f_2(x), f_3(x) \big)$. If A has rows $\mathbf{v}(x_1)$, $\mathbf{v}(x_2)$, $\mathbf{v}(x_3)$ and $\det(A) \neq 0$, then the functions are independent.

Answer: The last three apply to the given functions, while the others apply only to fixed vectors.

[20%] **Ch4(b)**: Give an example of a 3×4 matrix A of rank 2 with exactly one zero entry. Include an explanation of why your example has rank 2.

Answer: Let A initially be the matrix of all ones, then set $a_{11} = 0$.

[30%] **Ch4(c)**: Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\mathbf{x} = \mathbf{0}$. Apply the kernel theorem. This is theorem 2 in section 4.2 of Edwards-Penney. Then S is a subspace of $\mathcal{R}^3$.

[30%] **Ch4(d)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer: Independent. The rank of the augmented matrix of the three vectors is 3. Details should compute the rank or the number of pivot columns. The book has a determinant shortcut, which is OK, but it should be explained. There is no such thing as the determinant of a 4×3 matrix.

Place this page on top of all Ch4 work.