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Laplace Transform and its application for solving differential equations. Fourier and Z Transforms

Motivation. Transform methods are widely used in many areas of science and engineering. For example, transform methods are used in signal processing and circuit analysis, in applications of probability theory. The basic idea is to transform a function from its original domain into a transform domain where certain operations can be carried out more efficiently, carrying out the operation in the transform domain, and then carrying out an inverse transform of the result (from the transform domain to the original domain).

For example, the *convolution* operation of two functions of time t , $f(t)$ and $g(t)$ is defined as:

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau = \int_{-\infty}^{+\infty} f(t - \tau) \cdot g(\tau) d\tau$$

with τ a real number. The convolution in the time domain becomes multiplication in the Laplace or Fourier domain. As an application, consider a linear circuit with the impulse response $h(t)$ and with the signal $x(t)$ as input. Then the output of the linear circuit is $y(t) = x(t) * h(t)$. If $H(s)$, $X(s)$ and $Y(s)$ are the Laplace transforms of the impulse response, the input, and the output, respectively, then $Y(s) = H(s) \cdot X(s)$, as seen in Figure 1. Once we know $Y(s)$ we can apply the inverse Laplace Transform to obtain the response of the circuit, $y(t)$, function of time t .

Transform methods provide a bridge between the commonly used method of separation of variables and numerical techniques for solving linear partial differential equations. While in some ways similar to separation of variables, transform methods can be effective for a wider class of problems. Even when the inverse of the transform cannot be found analytically, numeric and asymptotic techniques now exist for their inversion.

Laplace Transform. Let \mathbb{R} be the field of real numbers and \mathbb{C} the field of complex numbers. Consider a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(t), t \in \mathbb{R}, t \geq 0$. Then the *Laplace Transform* of $f(t)$ is denoted as $\mathcal{L}[f(t)]$ and it is defined as $F(s)$ with $s \in \mathbb{C}$:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

The Laplace transform $F(s)$ typically exists for all complex numbers s such that $Re(s) > a$ where $a \in \mathbb{R}$ is a constant which depends on the behavior of $f(t)$.

The *Inverse Laplace Transform* is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds$$

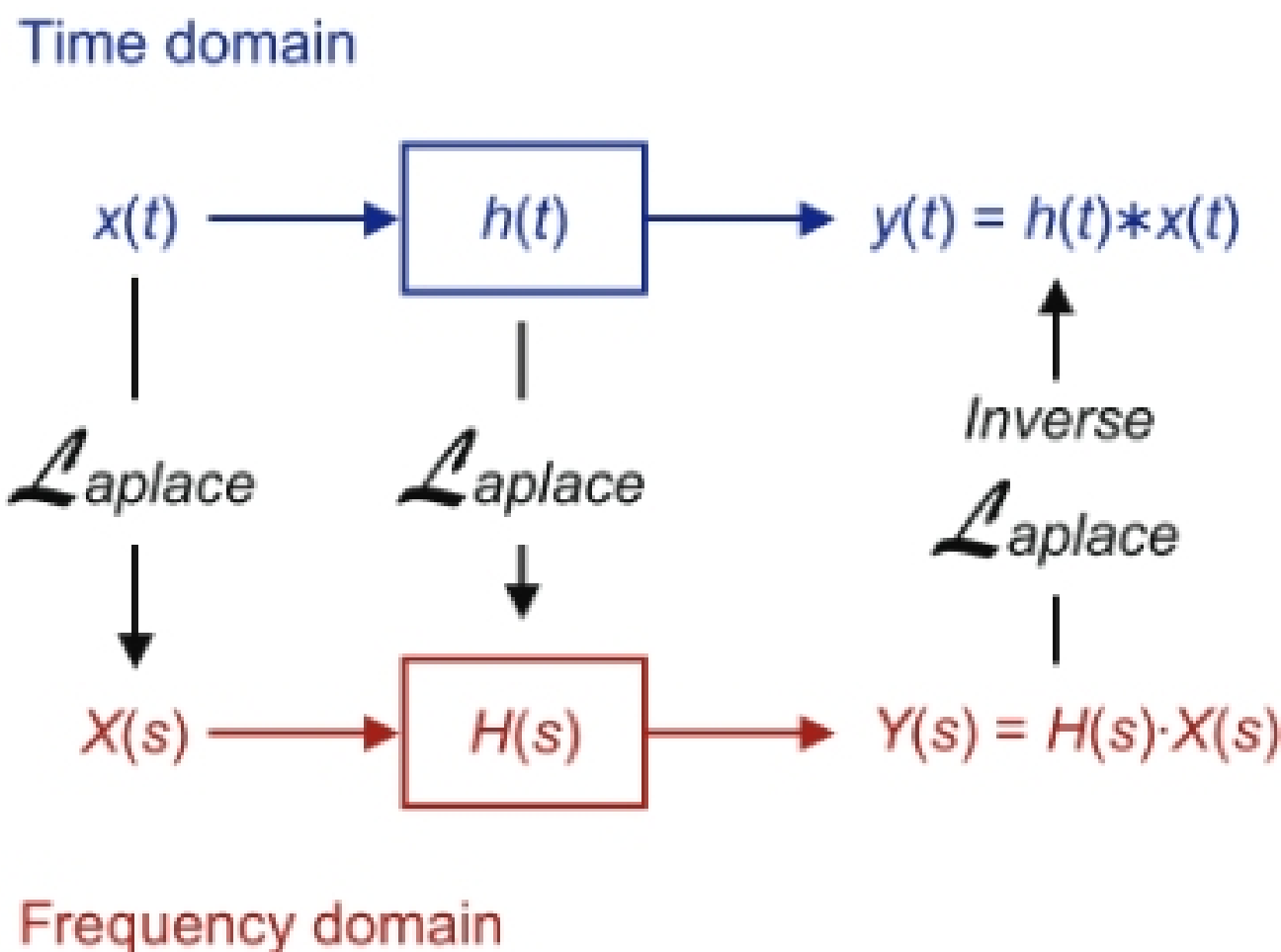


Figure 1: A circuit with the impulse response $h(t)$ and $x(t)$ as input. Then the output is $y(t) = x(t) * h(t)$. If $H(s)$, $X(s)$ and $Y(s)$ are respectively the Laplace transforms of the impulse response, the input, and the output, then $Y(s) = H(s) \cdot X(s)$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$ normally requiring $s_\rho > \text{Re}(s_\rho)$ for every singularity s_ρ of $F(s)$ and with $i = \sqrt{-1}$. If all singularities are in the left half-plane (in this case $\text{Re}(s_\rho) < 0$ for every s_ρ), then γ can be set to zero and the above inverse integral formula becomes identical to the inverse Fourier transform. This integral is known as the Fourier-Mellin integral.

The *Bilateral Laplace Transform* is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(t) dt.$$

In probability theory, the Laplace transform is defined by means of an expectation value. If X is a random variable with probability density function f_X , then the Laplace transform of f_X is given by the expectation:

$$(\mathcal{L}f_X)(s) = E[e^{-sX}].$$

By abuse of language, one often refers instead to this as the Laplace transform of the random variable X . Replacing the variable s by $-t$ gives the moment generating function of X . The Laplace transform has applications throughout probability theory, including first passage times of stochastic processes such as Markov chains and renewal theory.

Laplace Transforms of a few functions $f(t)$. In each case we start from the definition. For example, $f(t) = k$:

$$F(s) = \mathcal{L}[k] = \int_0^{\infty} e^{-st} k dt = -\frac{k}{s} [e^{-st}]_{t=0}^{t \rightarrow \infty} = \frac{k}{s}.$$

Now $f(t) = e^{\alpha t}$

$$F(s) = \mathcal{L}[e^{\alpha t}] = \int_0^{\infty} e^{-st} e^{\alpha t} dt = -\frac{1}{s-\alpha} \left[e^{-(s-\alpha)t} \right]_{t=0}^{t \rightarrow \infty} = \frac{1}{s-\alpha}.$$

Properties of the Laplace Transform. The properties of the Laplace Transform summarized in Table 1 can be derived easily starting from the definition.

Table 1: Properties of the Laplace Transform. The function $f(t)$ is assumed to be n -times differentiable, with n -th derivative of exponential type. Notations: $F(s) = \mathcal{L}[f(t)]$, $G(s) = \mathcal{L}[g(t)]$, $f^{(n)}$ is the n -th derivative of $f(t)$, $F^{(n)}(s)$ is the n -th derivative of $F(s)$, $u(t)$ is the Heaviside step function, $u(t) = \int_{-\infty}^t \delta(\tau) d\tau$ with δ the Dirac delta function, $(f * g)(t)$ is the convolution of $f(t)$ and $g(t)$, $\alpha, \beta \in \mathbb{R}$.

Property	Time-domain	s-domain
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Scaling	$f(\alpha t)$	$\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$
Frequency shifting	$e^{\alpha t} f(t)$	$F(s - \alpha)$
Time shifting	$f(t - \alpha) u(t - \alpha)$	$e^{-\alpha s} F(s)$
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
Frequency integration	$f(t)/t$	$\int_s^{\infty} F(r) dr$
Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
Integration	$\int_0^t f(\tau) d\tau$	$(u * f)(t) \frac{1}{s} F(s)$
Convolution	$(f * g)(t)$	$F(s) * G(s)$
Periodic function $f(t) = f(t + T)$	$f(t)$	$\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt$

For example, to prove linearity consider two functions $f(t)$ and $g(t)$ and their Laplace Transforms:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt \quad \text{and} \quad G(s) = \mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t) dt.$$

From the definition of the Laplace Transform it follows that

$$\mathcal{L}[f(t) + g(t)] = \int_0^{\infty} e^{-st} [f(t) + g(t)] dt = \int_0^{\infty} e^{-st} f(t) dt + \int_0^{\infty} e^{-st} g(t) dt = F(s) + G(s).$$

It is also easy to see that $F(0)$ represents the area under the curve $f(t)$:

$$F(s=0) = \int_0^{\infty} f(t) dt$$

The Laplace Transform can be expressed as:

$$\mathcal{L}[f(t)] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \dots$$