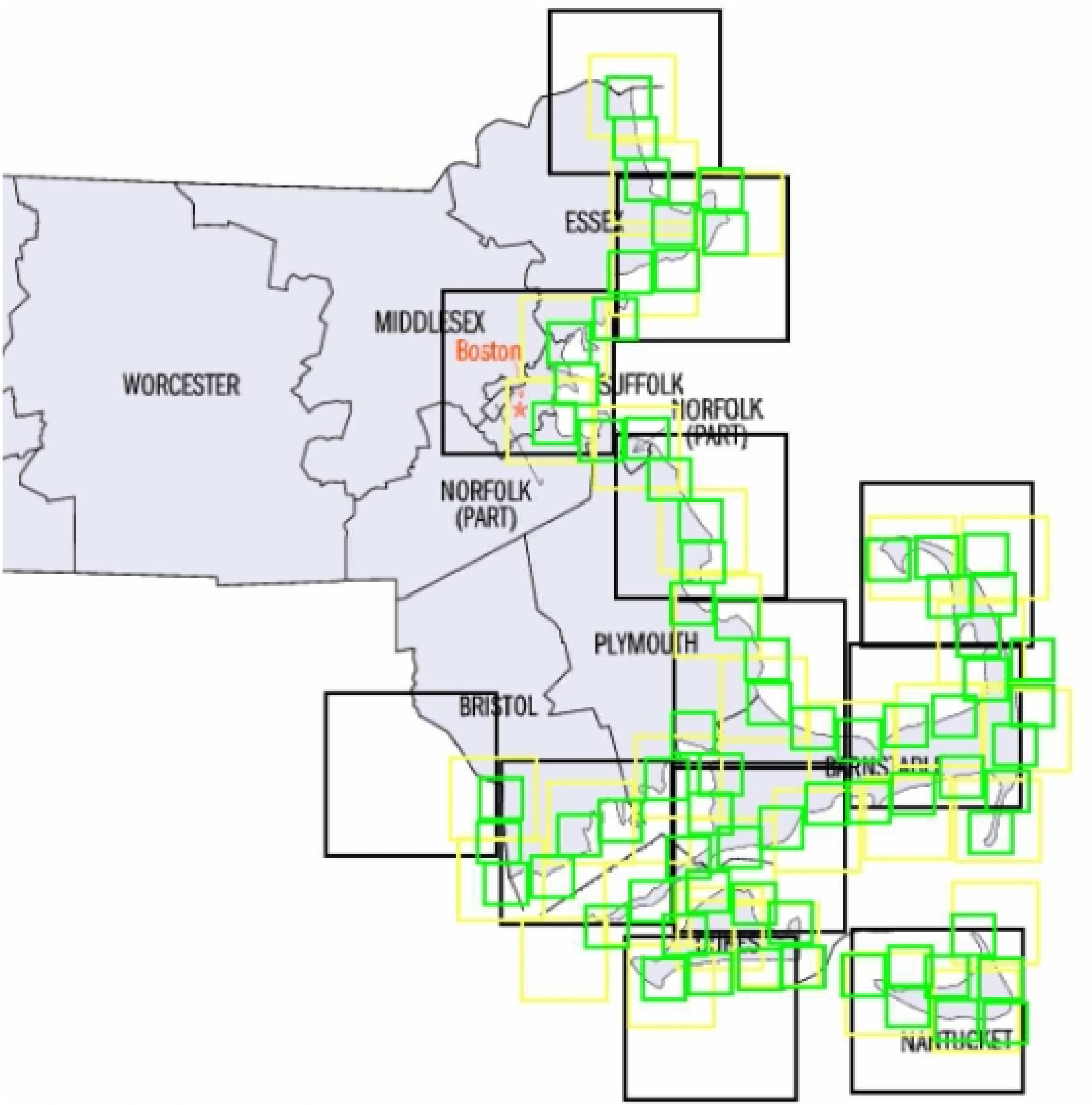
THE DIMENSION OF THE MASS COAST LINE

Math 21b, O.Knill

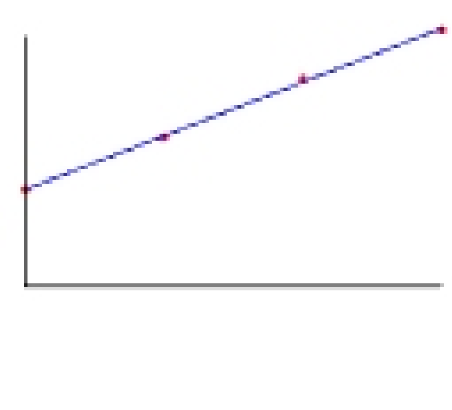
To measure the dimension of an object, one can count the number $f(n)$ of boxes of length $1/n$ needed to cover the object and see how $f(n)$ grows. If $f(n)$ grows like n^2, then the dimension is 2, if $n(k)$ grows like n, the dimension is 1. For fractal objects, like coast lines, the number of boxes grows like n^s for a number s between 1 and 2. The dimension is obtained by correlating $y_k = \log_2 f(k)$ with $x_k = \log_2(k)$. We measure:

> The Massachusetts coast line is a fractal of dimension 1.3.



We measure the data: $f(1) = 5$, $f(2) = 12$, $f(4) = 32$ and $f(8) = 72$. A plot of the data $(x_k, y_k) = (\log_2(k), \log_2 f(k))$ together with a **least square fit** can be seen to the right. The slope of the line is the dimension of the coast. It is about 1.295. In order to measure the dimension better, one would need better maps.

x_k	y_k
0	$\log_2(5)$
1	$\log_2(12)$
2	$\log_2(32)$
3	$\log_2(72)$



Finding the best linear fit $y = ax + b$ is equivalent to find the least square solution of the system

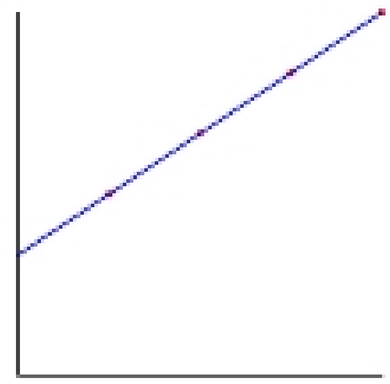
$$0a + b = \log_2(5), 1a + b = \log_2(12), 2a + b = \log_2(32), 3a + b = \log_2(72)$$

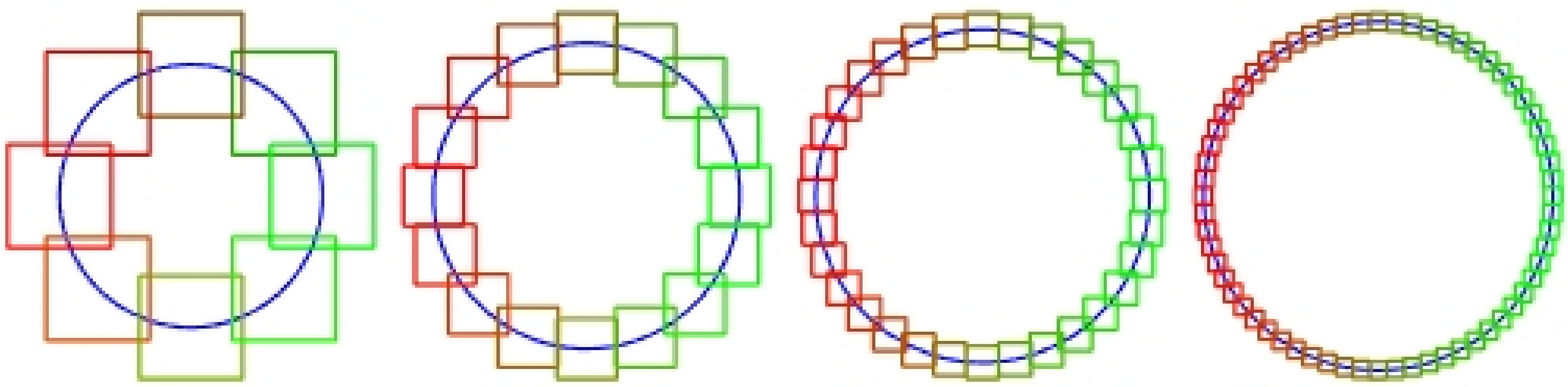
which is $A\vec{x} = \vec{b}$ with $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2.3 \\ 3.5 \\ 5 \\ 6 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$. We have $A^T A = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$, $(A^T A)^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 7 \end{bmatrix}/10$ and $B = (A^T A)^{-1} A^T = \begin{bmatrix} -3 & -1 & 1 & 3 \\ 7 & 4 & 1 & -2 \end{bmatrix}/10$. We get $= \begin{bmatrix} a \\ b \end{bmatrix} = B\vec{b} = \begin{bmatrix} 1.29 \\ 2.32 \end{bmatrix}$.

COMPARISON: THE DIMENSION OF THE CIRCLE

Let us compare this with a smooth curve. To cover a circle with squares of size $2\pi/2^n$, we need about $f(n) = 2^n$ to cover the circle. We measure that the circle is not a fractal. It has dimension 1.

x_k	y_k
3	$\log_2(8) = 3$
4	$\log_2(16) = 4$
5	$\log_2(32) = 5$
6	$\log_2(64) = 6$

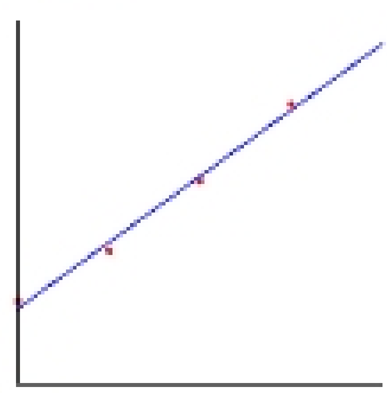


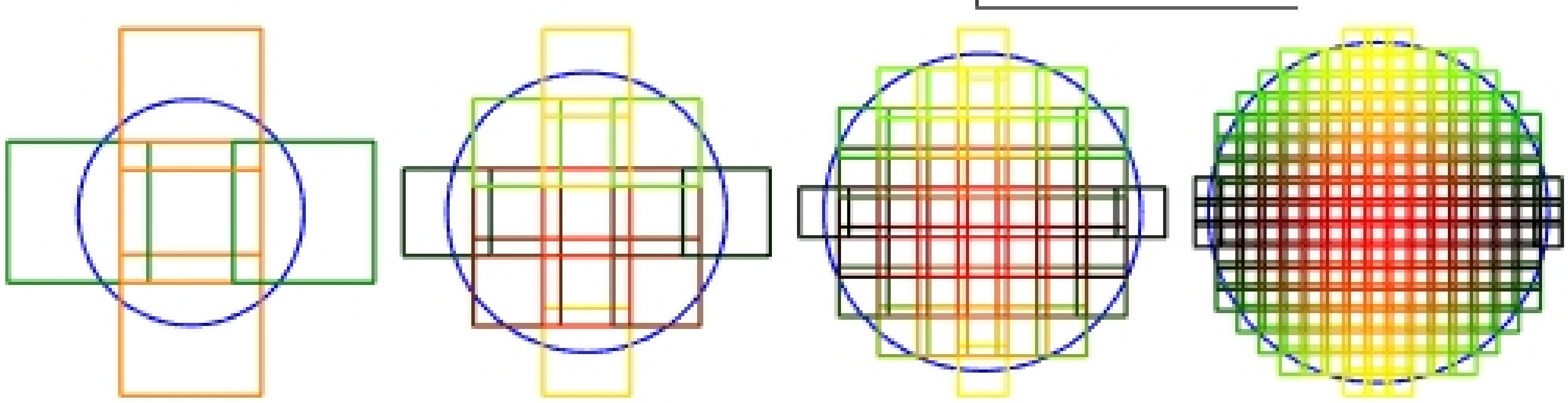


COMPARISON: THE DIMENSION OF THE DISK

For an other comparison, we take a disk. To cover the disk with squares of size $2\pi/2^n$, we need about $f(n) = 2^{2n}$ squares. We measure that the disk is not a fractal. It has dimension 2.

x_k	y_k
3	$\log_2(5) = 2.3$
4	$\log_2(13) = 3.7$
5	$\log_2(49) = 5.61$
6	$\log_2(213) = 7.73$





REMARKS

Calculating the dimension of coast lines is a classical illustration of "fractal theory". The coast of Brittain has been measured to have a dimension of 1.3 too. For natural objects, it is typical that measuring the length on a smaller scale gives larger results: one can measure empirically the dimension of mountains, clouds, plants, snowflakes, the lung etc. The dimension is an indicator how **rough** a curve or surface is.