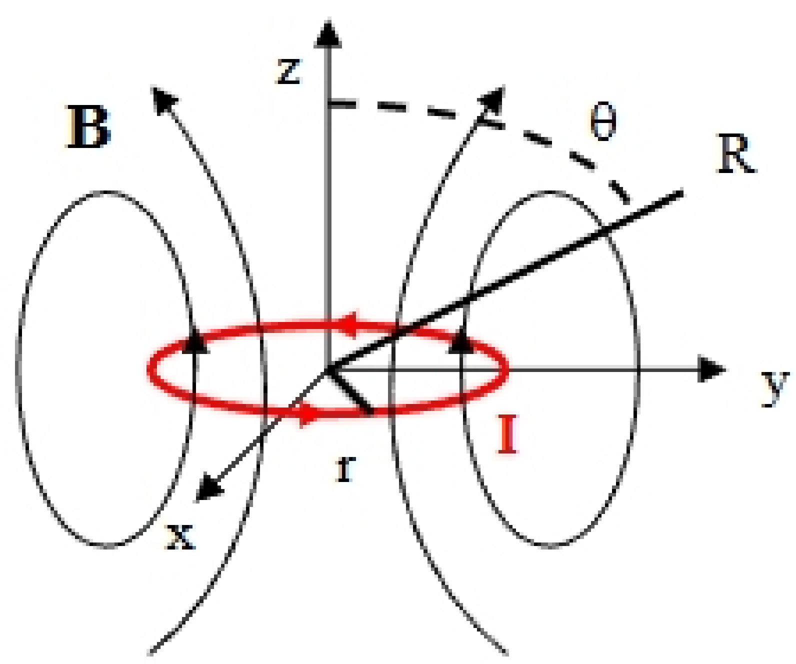
Magnetic Dipoles

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Magnetic Field of Current Loop



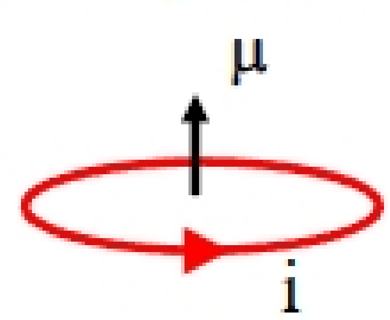
For distances $R \gg r$ (the loop radius), the calculation of the magnetic field does not depend on the shape of the current loop. It only depends on the current and the area (as well as R and θ):

$$\mathbf{B} = \begin{cases} B_r = 2|\boldsymbol{\mu}|\dfrac{\mu_0}{4\pi}\dfrac{\cos\theta}{R^3} \\ B_\theta = |\boldsymbol{\mu}|\dfrac{\mu_0}{4\pi}\dfrac{\sin\theta}{R^3} \end{cases}$$ where $\boldsymbol{\mu} = i\mathbf{A}$ is the magnetic dipole moment of the loop

Here i is the current in the loop, A is the loop area, R is the radial distance from the center of the loop, and θ is the polar angle from the Z-axis. The field is equivalent to that from a tiny bar magnet (a magnetic dipole).

We define the **magnetic dipole moment** to be a vector pointing out of the plane of the current loop and with a magnitude equal to the product of the current and loop area:

$$\boxed{\bar{\mu} \equiv i\,\bar{A}}$$



The area vector, and thus the direction of the magnetic dipole moment, is given by a right-hand rule using the direction of the currents.

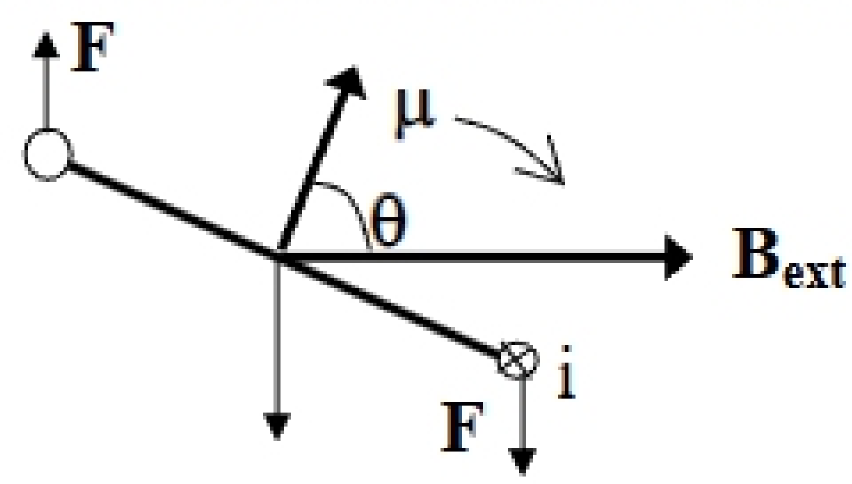
Interaction of Magnetic Dipoles in External Fields

Torque

By the $\mathbf{F} = i\mathbf{L}\times\mathbf{B}_{ext}$ force law, we know that a current loop (and thus a magnetic dipole) feels a torque when placed in an external magnetic field:

$$\boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}_{ext}$$

The direction of the torque is to line up the dipole moment with the magnetic field:



Potential Energy

Since the magnetic dipole wants to line up with the magnetic field, it must have higher potential energy when it is aligned opposite to the magnetic field direction and lower potential energy when it is aligned with the field.

To see this, let us calculate the work done by the magnetic field when aligning the dipole. Let θ be the angle between the magnetic dipole direction and the external field direction.

$$\begin{aligned} W &= \int \mathbf{F}\cdot d\mathbf{s} \\ &= \int |\mathbf{F}|\sin\theta ds = -\int r|\mathbf{F}|\sin\theta\, d\theta \quad (\text{where } ds = -rd\theta) \\ &= -\int |\mathbf{r}\times\mathbf{F}| d\theta \\ \Rightarrow W &= -\int |\boldsymbol{\tau}|\, d\theta \end{aligned}$$

Now the potential energy of the dipole is the negative of the work done by the field:

$$U = -W = \int \tau d\theta$$

The zero-point of the potential energy is arbitrary, so let's take it to be zero when θ=90°. Then:

$$U = +\int_{\pi/2}^{\theta} \tau d\theta = +\int_{\pi/2}^{\theta} \mu B \sin\theta' d\theta'$$

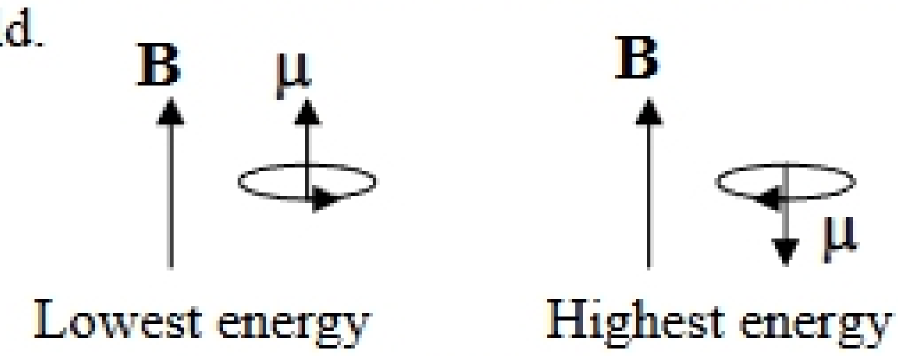
The positive sign arises because $\boldsymbol{\tau} \cdot d\boldsymbol{\theta} = -\tau d\theta$, τ and θ are oppositely aligned. Thus,

$$U = -\mu B \cos\theta \Big|_{\pi/2}^{\theta} = -\mu B \cos\theta$$

Or simply:

$$\boxed{U = -\boldsymbol{\mu} \cdot \mathbf{B}}$$

The lowest energy configuration is for $\boldsymbol{\mu}$ and $\mathbf{B}$ parallel. Work (energy) is required to re-align the magnetic dipole in an external B field.



The change in energy required to flip a dipole from one alignment to the other is

$$\boxed{\Delta U = 2\mu B}$$