Today's topics

- Relations
  - Kinds of relations
  - n-ary relations
  - Representations of relations
- Reading: Sections 7.1-7.3
- Upcoming
  - Minesweeper

Binary Relations

- Let A, B be any sets. A *binary relation* R from A to B, written (with signature) $R{:}A\times B$, or $R{:}A,B$, is (can be identified with) a subset of the set $A\times B$.
  - *E.g.*, let $< : \mathbf{N}\leftrightarrow\mathbf{N} :\equiv \{(n,m) \mid n < m\}$
- The notation $a\ R\ b$ or aRb means that $(a,b)\in R$.
  - *E.g.*, $a < b$ means $(a,b)\in <$
- If aRb we may say "a is related to b (by relation R)",
  - or just "a relates to b (under relation R)".
- A binary relation R corresponds to a predicate function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$ defined over the 2 sets A,B;
  - e.g., predicate "eats" $:\equiv \{(a,b) \mid$ organism a eats food $b\}$

Complementary Relations

- Let $R{:}A,B$ be any binary relation.
- Then, $\not{R}{:}A\times B$, the *complement* of R, is the binary relation defined by
  $\not{R} :\equiv \{(a,b) \mid (a,b)\notin R\} = (A\times B) - R$
- Note this is just $\overline{R}$ if the universe of discourse is $U = A\times B$; thus the name *complement*.
- Note the complement of $\not{R}$ is R.

Example: $\not< \; = \{(a,b) \mid (a,b)\notin <\} = \{(a,b) \mid \neg a<b\} = \geq$

Inverse Relations

- Any binary relation $R{:}A\times B$ has an *inverse* relation $R^{-1}{:}B\times A$, defined by
  $R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}$.

  E.g., $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, if R:People→Foods is defined by
  $a\ R\ b \Leftrightarrow a$ *eats* b, then:
  $b\ R^{-1}\ a \Leftrightarrow b$ *is eaten by* a. (Passive voice.)

Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on* the set A.
- *E.g.*, the "<" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- The (binary) *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a)|a \in A\}$.

Reflexivity

- A relation R on A is *reflexive* if $\forall a \in A$, aRa.
  - *E.g.*, the relation $\geq \,:\equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- A relation R is *irreflexive* iff its *complementary* relation $\overline{R}$ is reflexive.
  - Example: $<$ is irreflexive, because $\geq$ is reflexive.
  - Note "*irreflexive*" does **NOT** mean "*not reflexive*"!
    - For example: the relation "likes" between people is not reflexive, but it is not irreflexive either.
      - Since not everyone likes themselves, but not everyone *dislikes* themselves either!

Symmetry & Antisymmetry

- A binary relation R on A is *symmetric* iff $R = R^{-1}$, that is, if $(a,b) \in R \leftrightarrow (b,a) \in R$.
  - *E.g.*, **=** (equality) is symmetric. **<** is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *antisymmetric* if $\forall a \neq b, (a,b) \in R \rightarrow (b,a) \notin R$.
  - **Examples: <** is antisymmetric, "likes" is not.
  - **Exercise:** prove this definition of antisymmetric is equivalent to the statement that $R \cap R^{-1} \subseteq \mathbf{I}_A$.

Transitivity

- A relation R is *transitive* iff (for all a,b,c) $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.
- A relation is *intransitive* iff it is not transitive.
- Some examples:
  - "is an ancestor of" is transitive.
  - "likes" between people is intransitive.
  - "is located within 1 mile of" is… ?

Totality

- A relation $R:A\times B$ is *total* if for every $a\in A$, there is at least one $b\in B$ such that $(a,b)\in R$.
- If R is not total, then it is called *strictly partial*.
- A *partial relation* is a relation that *might* be strictly partial. (Or, it might be total.)
  - In other words, *all* relations are considered "partial."

Functionality

- A relation $R:A\times B$ is *functional* if, for any $a\in A$, there is *at most 1 b* $\in B$ such that $(a,b)\in R$.
  - "R is functional" $\Leftrightarrow$ $\forall a\in A: \neg\exists b_1\neq b_2\in B: aRb_1 \wedge aRb_2$.
  - Iff R is functional, then it corresponds to a partial function $R:A\to B$
    - where $R(a)=b \Leftrightarrow aRb$; *e.g.*
      - *E.g.*, The relation $aRb :\equiv$ "$a + b = 0$" yields the function $-(a) = b$.
- R is *antifunctional* if its inverse relation R^{-1} is functional.
  - Note: A functional relation (partial function) that is also antifunctional is an *invertible* partial function.
- R is a *total function* $R:A\to B$ if it is both functional and total, that is, for any $a\in A$, there is *exactly* 1 b such that $(a,b)\in R$. *I.e.*, $\forall a\in A: \neg\exists! b: aRb$.
  - If R is functional but not total, then it is a *strictly partial function*.
  - **Exercise**: Show that iff R is functional and antifunctional, and both it and its inverse are total, then it is a bijective function.

Lambda Notation

- The *lambda calculus* is a formal mathematical language for defining and operating on functions.
  - It is the mathematical foundation of a number of functional (recursive function-based) programming languages, such as LISP and ML.
- It is based on *lambda notation*, in which "$\lambda a: f(a)$" is a way to denote the function f *without ever assigning it a specific symbol*.
  - *E.g.*, $(\lambda x.\ 3x^2+2x)$ is a name for the function $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=3x^2+2x$.
- Lambda notation and the "such that" operator "$\ni$" can also be used to compose an expression for the function that corresponds to any given functional relation.
  - Let $R:A\times B$ be any functional relation on A,B.
  - Then the expression $(\lambda a: b \ni aRb)$ denotes the function $f:A\to B$ where $f(a) = b$ iff aRb.
    - E.g., If I write: $f :\equiv (\lambda a: b \ni a+b = 0)$, this is one way of defining the function $f(a)=-a$.

Composite Relations

- Let $R:A\times B$, and $S:B\times C$. Then the *composite* $S\circ R$ of R and S is defined as:
  $S\circ R = \{(a,c) \mid \exists b: aRb \wedge bSc\}$
- Note that function composition $f\circ g$ is an example.
- **Exer.**: Prove that $R:A\times A$ is transitive iff $R\circ R = R$.
- The n^{th} power R^n of a relation R on a set A can be defined recursively by:
  $R^0 :\equiv \mathbf{I}_A$; $R^{n+1} :\equiv R^n\circ R$ for all $n\geq 0$.
  - Negative powers of R can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.