
MATH 2300 - Calculus III - Section 8

Fall 2013

Homework 1 - Due: Friday, September 6

1. (Stewart, 54 of 12.3) A sphere, by definition, is all points of fixed distance (the radius) from a given point (the center). Show that all \vec{r} such that

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0 \tag{1}$$

form a sphere.

Solution:

$$\begin{aligned} (\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0 &\Leftrightarrow |\vec{r}|^2 - \vec{r} \cdot \vec{b} - \vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{b} = 0 \\ &\Leftrightarrow |\vec{r}|^2 - \vec{r} \cdot (\vec{a} + \vec{b}) + \vec{a} \cdot \vec{b} = 0 \\ &\Leftrightarrow |\vec{r}|^2 - 2\vec{r} \cdot \frac{\vec{a} + \vec{b}}{2} + \vec{a} \cdot \vec{b} = 0 \\ &\Leftrightarrow |\vec{r}|^2 - 2\vec{r} \cdot \frac{\vec{a} + \vec{b}}{2} + \frac{|\vec{a} + \vec{b}|^2}{4} - \frac{|\vec{a} + \vec{b}|^2}{4} + \vec{a} \cdot \vec{b} = 0 \\ &\Leftrightarrow |\vec{r}|^2 - 2\vec{r} \cdot \frac{\vec{a} + \vec{b}}{2} + \frac{|\vec{a} + \vec{b}|^2}{4} = \frac{|\vec{a} + \vec{b}|^2}{4} - \vec{a} \cdot \vec{b} \\ &\Leftrightarrow \left| \vec{r} - \frac{\vec{a} + \vec{b}}{2} \right|^2 = \frac{|\vec{a} + \vec{b}|^2}{4} - \vec{a} \cdot \vec{b} \end{aligned} \tag{2}$$

Notice that

$$\begin{aligned} \frac{|\vec{a} + \vec{b}|^2}{4} - \vec{a} \cdot \vec{b} &= \frac{|\vec{a}|^2}{4} + \frac{|\vec{b}|^2}{4} + \frac{\vec{a} \cdot \vec{b}}{2} - \vec{a} \cdot \vec{b} \\ &= \frac{|\vec{a}|^2}{4} + \frac{|\vec{b}|^2}{4} - \frac{\vec{a} \cdot \vec{b}}{2} \\ &= \frac{1}{4} (|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}) \\ &= \frac{1}{4} |\vec{a} - \vec{b}|^2 \geq 0. \end{aligned} \tag{3}$$

Therefore the given equation describes a circle of radius $\frac{1}{4} |\vec{a} - \vec{b}|^2$ and center $\frac{\vec{a} + \vec{b}}{2}$.

2. (Stewart, 61 and 62 of 12.3) Show that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ implies

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|. \tag{4}$$

Then use this to show that

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|. \tag{5}$$

Solution:

$$\begin{aligned} |\cos\theta| \leq 1 &\Rightarrow |\vec{a}||\vec{b}|\cos\theta \leq |\vec{a}||\vec{b}| \\ &\Rightarrow |\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|. \end{aligned} \tag{6}$$

For the second part:

$$\begin{aligned} |\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} \\ &\leq |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \\ &= (|\vec{a}| + |\vec{b}|)^2. \end{aligned} \tag{7}$$

Now take square root of both positive sides.

3. Show that

$$|\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2. \tag{8}$$

Solution:

$$\begin{aligned} |\vec{a} \cdot \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta \\ |\vec{a} \times \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta \end{aligned} \tag{9}$$

Therefore

$$|\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 (\cos^2 \theta + \sin^2 \theta) = |\vec{a}|^2 |\vec{b}|^2. \tag{10}$$

4. Show that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}. \tag{11}$$

Solution: Both sides of the equation are vectors. It suffices to show that their components are equal. We check here the first component only. The other two are similar.

The last two components of $\vec{b} \times \vec{c}$ are

$$-b_1c_3 + b_3c_1, \quad b_1c_2 - b_2c_1. \tag{12}$$

Therefore the first component of $\vec{a} \times (\vec{b} \times \vec{c})$ is

$$a_2(b_1c_2 - b_2c_1) - a_3(-b_1c_3 + b_3c_1). \tag{13}$$

On the other hand the first component of $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ is

$$\begin{aligned} (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 &= a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1 \\ &= a_2(b_1c_2 - b_2c_1) - a_3(b_1c_3 - b_3c_1), \end{aligned} \tag{14}$$

clearly the same as (??).

5. Given vectors $\vec{a} \neq 0$ and \vec{b} explain why there exists vector \vec{r} such that \vec{r} satisfies both

$$\vec{r} \cdot \vec{a} = 1, \quad \vec{r} \times \vec{a} = \vec{b}. \tag{15}$$

Solution: The first equation is the equation of a plane perpendicular to the vector \vec{a} .

The second equation is the equation of a line parallel to the vector \vec{a} .

Any such line has intersection with any such plane at a single point.

6. Stewart, section 12.4, problems 44, 48, 49, 53.

44 (a) Solution:

All \vec{r} 's satisfying $(1, 2, 1) \times \vec{r} = (3, 1, -5)$ form a line parallel to $(1, 2, 1)$, as discussed in class. This line will be completely determined if we find a point on it. As $(3, 1, -5)$ is the projection of any point on this line to the direction perpendicular to $(1, 2, 1)$ multiplied by $|(1, 2, 1)| = \sqrt{6}$, we divide $(3, 1, -5)$ by this length to get

$$\left(\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{5}{\sqrt{6}} \right). \quad (16)$$

This is the rotation by $\pi/2$ around $(1, 2, 1)$ of a vector on the given line and this vector is given by

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix} = \left(\frac{11}{6}, -\frac{8}{6}, \frac{5}{6} \right).$$

Indeed,

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ \frac{11}{6} & -\frac{8}{6} & \frac{5}{6} \end{vmatrix} = (3, 1, -5). \quad (17)$$

44 (a) Solution: $(3, 1, 5)$ is not perpendicular to $(1, 2, 1)$.

48 Solution:

$$\vec{a} \times \vec{b} = \vec{a} \times (-\vec{a} - \vec{c}) = \vec{a} \times (-\vec{a}) + \vec{a} \times (-\vec{c}) = -\vec{a} \times \vec{a} - \vec{a} \times \vec{c} = -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}. \quad (18)$$

Similarly,

$$\vec{a} \times \vec{b} = (-\vec{b} - \vec{c}) \times \vec{b} = -\vec{c} \times \vec{b} = \vec{b} \times \vec{c}. \quad (19)$$

49 Solution:

$$\begin{aligned} (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) &= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b} \\ &= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b} \\ &= -\vec{b} \times \vec{a} + \vec{a} \times \vec{b} \\ &= \vec{a} \times \vec{b} + \vec{a} \times \vec{b} \\ &= 2(\vec{a} \times \vec{b}). \end{aligned} \quad (20)$$

53 Solution:

(a) No. \vec{b} and \vec{c} could be on the same plane perpendicular to \vec{a} .

(b) No. \vec{b} and \vec{c} could be on the same line parallel to \vec{a} .

(c) Yes. Both \vec{b} and \vec{c} are the intersection of the same line with the same plane.