MATH 2300 - Calculus III - Section 8

Fall 2013

Homework 2 - Due Friday, September 20

1. (Rotations on the plane.) Apply the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

on a point of P of coordinates (x, y) of the plane to get a new point P' with coordinates (x', y'). Show that the distance from the origin has not changed. Show that $\overrightarrow{OP'}$ is rotation of $\overrightarrow{OP}$ by angle θ around the origin (i.e. the length does not change and the vectors form angle θ).

Solution:

$$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2 + y^2. \tag{1}$$

$$(x, y) \cdot (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = (x^2 + y^2)\cos\theta = \left|\vec{OP}\right|\left|\vec{OP'}\right|\cos\theta. \tag{2}$$

2. (Some matrices are nothing more than vectors.) Apply the matrix

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{3}$$

on a vector $\vec{r} = (x, y, z)$. Show that the result is

$$\vec{\Omega} \times \vec{r}, \tag{4}$$

where $\vec{\Omega}$ is the vector $\vec{\Omega} = (\omega_1, \omega_2, \omega_3)$.

Solution: Straight forward calculation.

3. (Arc length reparametrization.) Let $\vec{c}(t),\ t \in [a, b]$ be a curve in 3 space. Define the function of t

$$s(t) = \int_a^t \left|\vec{c}\,'(\tau)\right|\, d\tau. \tag{5}$$

Assume that the velocity of c does not vanish at any time.

(a) Show that s is a strictly increasing function of t.

(b) If L is the total length of $\vec{c}$, show that s takes the interval $[a, b]$ to the interval $[0, L]$ in an one-to-one manner.

(c) If $s^{-1} : [0, L] \to [a, b]$ is the inverse function of s, explain why $\vec{c}(s^{-1}(\alpha)),\ \alpha \in [0, L]$ is another way of describing the same curve (i.e. a re-parametrization of $\vec{c}$).

(d) Show that the length of the curve from its beginning to the point $\vec{c}(s^{-1}(\alpha))$ is α.

Solution

(a) By the fundamental theorem of calculus,

$$s'(t) = |\vec{c}'(t)| > 0. \tag{6}$$

(b) $s(a) = 0, s(b) = L$ by definition of s. Strictly increasing implies $1-1$.

(c) The curve is precisely the points of the form $\vec{c}(t), t \in [a,b]$. But since s is one to one and onto $[0, L]$, every t is of the form $s^{-1}(\alpha)$ for some $\alpha \in [0, L]$.

(d) If you set $t = s^{-1}(\alpha)$, then $s(t) = \alpha$. At the same time, the length of the curve at time t is $s(t)$, by definition of s.

4. (A calculation from classical mechanics.)

(a) Show that the only vector that is perpendicular to all vectors is the $\vec{0}$.

(b) Show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c}). \tag{7}$$

(c) Show that if two curves $\vec{\gamma}_1, \vec{\gamma}_2$ satisfy

$$\vec{\gamma}_1{}'(t) \cdot \vec{v} \times \vec{\gamma}_2(t) + \vec{\gamma}_1(t) \cdot \vec{v} \times \vec{\gamma}_2{}'(t) = 0 \tag{8}$$

for any constant vector $\vec{v}$, then

$$\vec{\gamma}_1(t) \times \vec{\gamma}_2(t) \tag{9}$$

is constant in t.

Solution:

(a) If $\vec{r}$ is perpendicular to all $\vec{v}$, then $\vec{r} \cdot \vec{v} = 0$, for all $\vec{v}$. Taking $\vec{v} = \vec{r}$ gives $|\vec{r}|^2 = 0$, therefore $\vec{r} = 0$.

(b) Show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c}) \tag{10}$$

by direct calculation left and right.

(c)

$$\begin{aligned} &\vec{\gamma}_1{}'(t) \cdot \vec{v} \times \vec{\gamma}_2(t) + \vec{\gamma}_1(t) \cdot \vec{v} \times \vec{\gamma}_2{}'(t) \\ &= -\vec{v} \cdot \vec{\gamma}_1{}'(t) \times \vec{\gamma}_2(t) - \vec{v} \cdot \vec{\gamma}_1(t) \times \vec{\gamma}_2{}'(t) && \text{by (b)} \\ &= -\vec{v} \cdot \left(\vec{\gamma}_1{}'(t) \times \vec{\gamma}_2(t) + \vec{\gamma}_1(t) \times \vec{\gamma}_2{}'(t)\right) && \text{by distributive property of dot product} \\ &= -\vec{v} \cdot (\vec{\gamma}_1(t) \times \vec{\gamma}_2(t))\,'. && \text{by cross product rule} \end{aligned} \tag{11}$$

You are given that this is 0 for all $\vec{v}$.
Therefore $(\vec{\gamma}_1(t) \times \vec{\gamma}_2(t))\,' = 0$ by (a).
Therefore $\vec{\gamma}_1(t) \times \vec{\gamma}_2(t)$ is constant.

5. Recall that the speed of a curve $\vec{r}(t)$ at t is the length of its velocity vector at t. Use Calculus I to find the time of minimum speed and the minimum speed of the curve $\vec{r}(t) = (t^2,\ 5t,\ t^2 - 16t),\ t > 0$. Does the curve reach a maximum speed?

Solution: Standard 1-dimensional Calculus on $f(t) = |\vec{r}(t)|$, or on $g(t) = |\vec{r}(t)|^2$.

6. Give a parametrization for a spiral curve that starts from $(2, 0, 0)$, spirals three times on the cylinder $x^2+y^2 = 4$, and ends up at $(2, 0, 9)$.

Solution: $\left(2\cos t, 2\sin t, \frac{9}{6\pi}\right)$, $t \in [0, 6\pi]$, or $\left(2\cos 3t, 2\sin 3t, \frac{9}{2\pi}\right)$, $t \in [0, 2\pi]$, or ...