MATH 2300 Fall 2013

Homework 3 - Solutions

1. **Solution:**

The point (2, 1, 3) happens on r_1 when $t = 0$ and on r_2 when $t = 1$. The tangent are

$$\begin{aligned} r_1'(t) &= (\ 3,\ -2t,\ -4+2t\ ) \Rightarrow r_1'(0) = (\ 3,\ 0,\ -4\ ) \\ r_2'(t) &= (\ 2t,\ 6t^2,\ 2\ ) \Rightarrow r_2'(1) = (\ 2,\ 6,\ 2\ ) \end{aligned} \tag{1}$$

Then both (3, 0, −4) and (2, 6, 2) are tangent to the surface at the point (2, 1, 3). Therefore both (3, 0, −4) and (2, 6, 2) are on the tangent plane to the surface at (2, 1, 3).

We want therefore the equation of the plane that goes through (2, 1, 3) and is spanned by (3, 0, −4) and (2, 6, 2).

The cross product

$$(\ 3,\ 0,\ -4\ ) \times (\ 2,\ 6,\ 2\ ) = \begin{vmatrix} i & j & k \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = (24, -14, 18) \tag{2}$$

is perpendicular to the plane and therefore $(12, -7, 9)$ also is. The equation of the plane then is

$$(12, -7, 9) \cdot (\vec{r} - (\ 2,\ 1,\ 3\ )) = 0. \tag{3}$$

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2. **Solution:** i. By chain rule,

$$\frac{d}{dt} f(tx,\ ty) = \nabla f(tx,\ ty) \cdot (x,\ y) = x\frac{\partial f}{\partial x}(tx, ty) + y\frac{\partial f}{\partial y}(tx, ty) \tag{4}$$

and this must equal $nt^{n-1} f(x,y)$. Now evaluate at $t = 1$.

ii.

Differentiate (4) again with respect to t to get

$$\begin{aligned} & x\nabla\left(\frac{\partial f}{\partial x}\right)(tx, ty)\cdot(x,y) + y\nabla\left(\frac{\partial f}{\partial y}\right)(tx,ty)\cdot(x,y) \\ =& \left(x\frac{\partial^2 f}{\partial x^2}(tx,ty) + x\frac{\partial^2 f}{\partial x \partial y}(tx,ty)\right)\cdot(x,y) + \left(y\frac{\partial^2 f}{\partial x\partial y}(tx,ty) + y\frac{\partial^2 f}{\partial y^2}(tx,ty)\right)\cdot(x,y) \\ =& x^2\frac{\partial^2 f}{\partial x^2}(tx,ty) + 2xy\frac{\partial^2 f}{\partial x\partial y}(tx,ty) + y^2\frac{\partial^2 f}{\partial y^2}(tx,ty) \end{aligned} \tag{5}$$

and this must be $n(n-1)t^{n-2} f(x,y)$. Again evaluate at $t = 1$.

iii.

Notice that

$$\begin{aligned}
\frac{\partial f}{\partial x}(tx,ty) &= \nabla f(tx,ty)\cdot(1,0) \\
&= \left(D_{(1,0)}f\right)(tx,ty) \\
&= \lim_{h\to 0}\frac{f(tx+h,ty)-f(tx,ty)}{h} \\
&= \lim_{h\to 0}\frac{f(t(x+h/t),ty)-f(tx,ty)}{h} \\
&= \lim_{h\to 0}\frac{t^n f((x+h/t),y)-t^n f(x,y)}{h} \\
&= t^{n-1}\lim_{\frac{h}{t}\to 0}\frac{f((x+h/t),y)-f(x,y)}{h/t} = t^{n-1}\frac{\partial f}{\partial x}(x,y).
\end{aligned} \tag{6}$$

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3. i. **Solution:** i.

The functions should be standard the *sin* on half period marching to the left and to the right.

Calculate via chain rule

$$\begin{aligned}
\frac{\partial g}{\partial t} &= f'(x+at)a - f'(x-at)a, \\
\frac{\partial^2 g}{\partial t^2} &= f''(x+at)a^2 + f''(x-at)a^2, \\
\frac{\partial g}{\partial x} &= f'(x+at) + f'(x-at) \\
\frac{\partial^2 g}{\partial x^2} &= f''(x+at) + f''(x-at).
\end{aligned} \tag{7}$$

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4. **Solution:**

No: For any $\vec{v}$ with $D_{\vec{v}}f(\vec{x}_0) = \nabla f(\vec{x}_0)\cdot\vec{v} > 0$ we have for $D_{-\vec{v}}f(\vec{x}_0) = -\nabla f(\vec{x}_0)\cdot\vec{v} < 0$.

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5. **Solution:**

If $f(x,y) = x$ then $D_{(1,0)}f(\vec{x}) = \nabla f(\vec{x})\cdot(1,0) = (1,0)\cdot(1,0) = 1 > 0$.

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6. **Solution:**

If $\vec{a} = (a_1,a_2,a_3)$ then $\nabla f(x,y,z) = (a_1,a_2,a_3) = \vec{a}$. And $D_{\vec{v}}f(\vec{x}) = \nabla f(x,y,z)\cdot\vec{v} = \vec{a}\cdot\vec{v}$.

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7. **Solution:**

i.

$\frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \frac{d}{dt}\vec{r}(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\frac{d}{dt}\vec{r}(t)n = 2\frac{d}{dt}\vec{r}(t)\cdot\vec{r}(t) = 0$, therefore $\vec{r}\,'(t)\perp\vec{r}(t)$.

ii.

Use for $\vec{r}(t), t \in [0,1]$ any half-circle of radius a from $(0,0,a)$ to $(0,0-a)$ to calculate

$$\begin{aligned} \frac{d}{dt} f(\vec{r}(t_0)) &= \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \text{ by chain rule} \\ &= k(t) r(t) \cdot \vec{r}'(t) \text{ by assumption on } f \\ &= 0 \text{ by part i.} \end{aligned} \tag{8}$$
