

# ME 201/MTH 281/ME 400/CHE 400

## ASSIGNMENT #8 2010

Assignments handed in by 6 PM on Wednesday Nov. 3 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday Nov. 4 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Nov. 4. This is the last homework assignment before Exam #2 on Nov. 11.

### LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
V. SEPARATION OF VARIABLES, PART 2		
5.4 Organ Pipe Acoustics	W Oct 27	---
5.5 Standing Acoustic Modes in Three Dimensions	Th Oct 28	7.3,7.4
VI. UNBOUNDED DOMAINS		
6.1 Fourier Integral	F Oct 29	10.1-10.3
6.2 Fourier Transform	M Nov 1	10.3,10.4

### PROBLEMS

1) (35 points) A thin rectangular steel plate has horizontal dimensions  $a$  by  $b$ , with a thickness  $h$  in the vertical, where  $h$  is much smaller than  $a$  and  $b$ . The plate is simply supported all around its edges. (Simply supported means that the edges are fixed in position, but are free to rotate in response to moments.) The object of this problem is to find the normal modes of vibration of the plate, when the vibratory motion is all in the vertical direction. For small amplitudes, the vertical displacement  $w(x,y,t)$  satisfies the equation

$$\frac{\partial^2 w}{\partial t^2} = -\sigma \nabla^4 w \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad , \quad \text{where } \nabla^4 w = \nabla^2(\nabla^2 w) \quad .$$

Here  $\sigma$  is a positive constant given by  $\sigma = \frac{E h^2}{12 \rho (1 - \nu^2)}$ , where  $E$  is Young's modulus,  $\rho$  is the density, and  $\nu$  is Poisson's ratio. The boundary conditions for the simply supported plate are

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{on } x = 0 \quad \text{and } x = a, \quad \text{and} \quad w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{on } y = 0 \quad \text{and } y = b .$$

(a) (15 points) Find the frequencies of the normal modes for the plate. Do this by looking for modes in the form  $\cos(\omega t)F(x)G(y)$ . Do not try to find  $F$  and  $G$  by separation of variables. You will just get bogged down. Instead, think about functions you already know which (1) satisfy the boundary conditions and (2) are eigenfunctions of  $\partial^2 / \partial x^2$  and  $\partial^2 / \partial y^2$ .

(b) (15 points) Find the four modes with the lowest four frequencies when  $b = 2a$ . Describe the nodal lines (lines of zero amplitude) for each of those modes.

(c) (5 points) For steel,  $E = 30 \times 10^6 \text{ lb/in}^2$ ,  $\nu = 0.3$  (dimensionless), and  $\rho = 490 \text{ lb}_m/\text{ft}^3$ . If the plate dimensions are  $a = 0.5 \text{ ft}$ ,  $b = 1 \text{ ft}$ , and  $h = 0.01 \text{ ft}$ , what are the frequencies, in Hz, of the four lowest modes? (Be careful with the units!)

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(2) (25 points) ) Consider a vibrating membrane which has a tension  $T$  (force per unit length), and a mass per unit area  $\sigma$ . The membrane has a vertical displacement  $\Psi(x,y,t)$ , where  $x, y$  are coordinates in the plane of the membrane and  $t$  is time. It can be shown that the equation of motion for small displacements is

$$\frac{\partial^2 \Psi}{\partial t^2} = C^2 \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right), \quad \text{where } C^2 = \frac{T}{\sigma}.$$

The membrane occupies the rectangular region  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ , and the displacement  $\Psi$  is zero everywhere along the boundary of this region. If  $a = 2$  m,  $b = 3$  m,  $T = 10^3$  N/m, and  $\sigma = 0.1$  kg/m<sup>2</sup>, estimate the number of modes with frequencies between 2000 and 4000 Hz.

(3) (20 points) This problem deals with the function  $f(x) = e^{-|x|} \cos(x)$  and with its Fourier transform.

(a) (7 points) Find the Fourier transform  $\tilde{f}(k)$  of  $f(x)$  by using Mathematica to evaluate the integral defining the transform.

(b) (6 points) Use the definition of the Fourier transform to show that in general

$\int_{-\infty}^{\infty} f(x) dx = \tilde{f}(0)$ . Verify this result for the particular  $f(x)$  of this problem.

(c) (7 points) Verify Parseval's theorem for this transform pair by using Mathematica to evaluate both of the relevant integrals. (Reminder: Parseval's Theorem says that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk.$$

(4) (20 points) In this problem, you will apply the Fourier Transform to the solution of a simple ordinary differential equation for a function  $y(x)$ . The equation is

$$y'' - 2y = e^{-|x|} \cos(x), \quad \text{with } y \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

(a) (10 points) Take the Fourier transform of the equation and find  $\tilde{y}(k)$ , the Fourier transform of  $y(x)$ .

(b) (10 points) Invert the transform to find  $y(x)$ . Do this by using Mathematica to evaluate the inversion integral.

### CHALLENGE PROBLEM

This problem connects with our earlier discussion of the guitar string, but also with the standing modes considered in Chapter 5. The new feature here is the damping of standing wave modes. Although it is possible to do this for acoustic modes, it is complicated, because one has to include viscosity and heat conduction, and compliance at a wall. To keep the calculation simple and to work in one space dimension, we go back to the vibrating string. We now assume that at each point of the string, there is a damping force proportional to the string velocity, giving

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**CHALLENGE PROBLEM (continued)**

rise to a term in the equation proportional to the velocity. Then the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} - \varepsilon \frac{\partial y}{\partial t}, \quad 0 < x < L \text{ and } t > 0,$$

where  $\varepsilon$  is a positive constant. We take the usual boundary conditions of zero displacement at the ends:  $y(0,t) = y(L,t) = 0$ . In the absence of damping ( $\varepsilon = 0$ ), we showed earlier that the  $n$ th mode of free vibration has the form

$$y_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin\left(\frac{n\pi x}{L}\right), \text{ where } \omega_n = \frac{n\pi C}{L}.$$

If the damping is small (in a sense which has to be made precise), we would expect to get the above modes modulated by a slow exponential damping. That is the result that you will derive in this problem. If we have the opposite extreme of large damping, then the lower modes will not be oscillatory at all. That is also a result you will derive in this problem.

(a) (25 points) Look for modes of the form  $y_n(x,t) = F_n(t) \sin(n\pi x / L)$ . Find  $F_n(t)$ .

(b) (25 points) Show that for  $\varepsilon < \frac{2\pi C}{L}$ , all of the modes, though damped, have some oscillatory

behavior. Show that for  $\varepsilon > \frac{2\pi C}{L}$ , one or more modes, including the lowest mode, will have purely exponential behavior with decaying exponentials.

(c) (25 points) Show that for  $\varepsilon \ll \frac{2\pi C}{L}$ , a valid approximation is

$$F_n(t) = e^{-\frac{1}{2}\varepsilon t} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)],$$

where  $\omega_n = (n\pi C / L)$  is the undamped frequency of the  $n$ th mode.

(d) (25 points) Consider a vibrating guitar string – namely the high E string in the fundamental mode (the frequency is 330 Hz). Suppose it is observed that the amplitude of the free vibration drops by a factor of  $\frac{1}{2}$  in 8 seconds. Estimate the value of  $\varepsilon$  for this string. Verify that the inequality  $\varepsilon \ll \frac{2\pi C}{L}$  is satisfied (this is necessary to use the approximation of part (c)).