MA123, Supplement: Exponential and logarithmic functions (pp. 315-319) **Date:** ___________

`Chapter Goals:`

- Review the properties of exponential and logarithmic functions.
- Learn how to differentiate exponential and logarithmic functions.
- Learn about exponential growth or decay phenomena.

`Assignments:` `Assignment 10` `Assignment 11`

Quick review

Exponential notation: If a is any real number and n is a positive integer, then the **n-th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \ \dots \ \cdot a}_{n \text{ times}}$$

The number a is called the **base** whereas n is called the **exponent**.

The first and second laws of exponents below allow us to define a^n for any integer n.

Now, we want to define, for instance, $a^{1/3}$ in a way that is consistent with the laws of exponents. We would like:

$$\left(a^{1/3}\right)^3 = a^{(1/3)3} = a^1 = a; \qquad \text{thus} \qquad a^{1/3} = \sqrt[3]{a}$$

So, by the definition of nth root, we have:

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

Definition of rational exponents:
For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \qquad \text{or equivalently}$$

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

If n is even we require that $a \geq 0$.

In the table below the bases a and b are real numbers ($\neq 0$ if needed) and the exponents x and y are rational numbers.

Laws of exponents:

1. $a^0 = 1$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $a^x a^y = a^{x+y}$
4. $\dfrac{a^x}{a^y} = a^{x-y}$
5. $(a^x)^y = a^{xy}$
6. $(ab)^x = a^x b^x$
7. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Now, let $a > 0$ be a positive number with $a \neq 1$. Thus far a^x is defined for x a rational number. So, what does, for instance, $5^{\sqrt{2}}$ mean? When x is irrational, we successively approximate x by rational numbers. For instance, as

$$\sqrt{2} \approx 1.41421\dots$$

we successively approximate $5^{\sqrt{2}}$ with

$$5^{1.4}, \quad 5^{1.41}, \quad 5^{1.414}, \quad 5^{1.4142}, \quad 5^{1.41421}, \dots$$

In practice, we simply use our calculator and find out

$$5^{\sqrt{2}} \approx 9.73851\dots$$

▶ **Exponential functions:**
Let $a > 0$ be a positive number with $a \neq 1$. The **exponential function with base** a is defined by

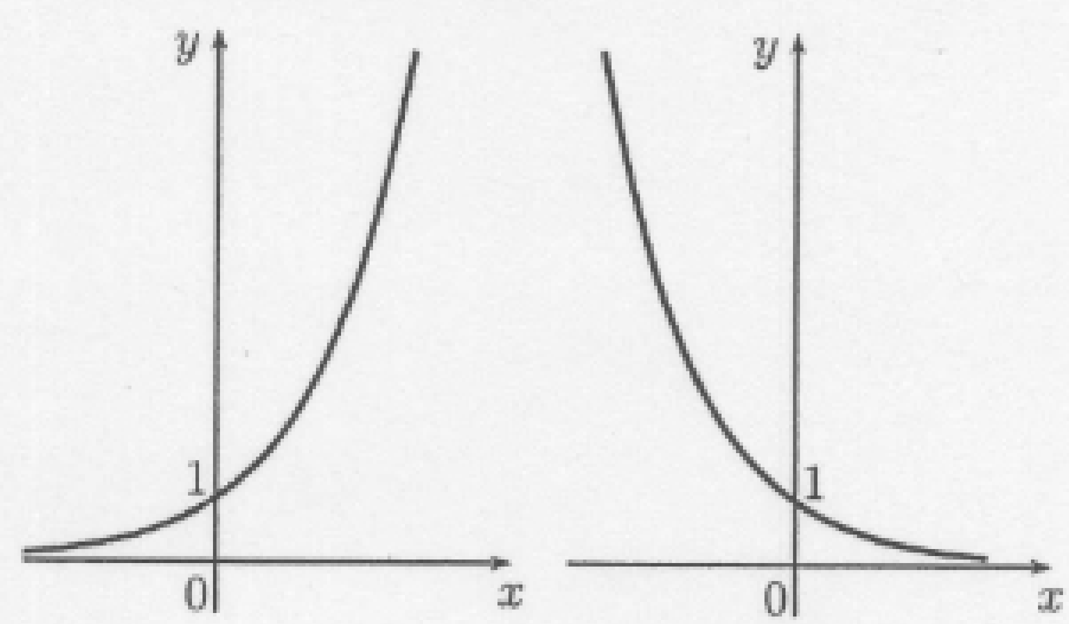
$$f(x) = a^x$$

for all real numbers x.

Graphs of exponential functions:
The **exponential function**

$$f(x) = a^x \qquad (a > 0, \ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

The most important base is the number denoted by the letter e. The number e is defined as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

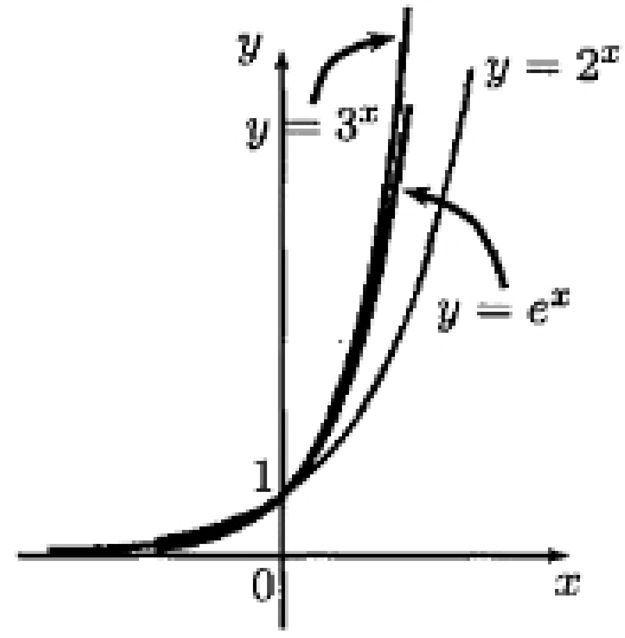
The natural exponential function:

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

► **Logarithmic functions:** Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the *horizontal line test*. Thus, it has an inverse function. The inverse function $f^{-1}(x)$ is called the *logarithmic function with base* a and is denoted by $\log_a x$.

Definition: Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \quad \iff \quad a^y = x.$$

In other words, $\log_a x$ is the exponent to which the base a must be raised to give x.

Properties of logarithms:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Since logarithms are 'exponents', the laws of exponents give rise to the laws of logarithms:

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

Laws of logarithms:

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C \log_a A$.

Change of base:

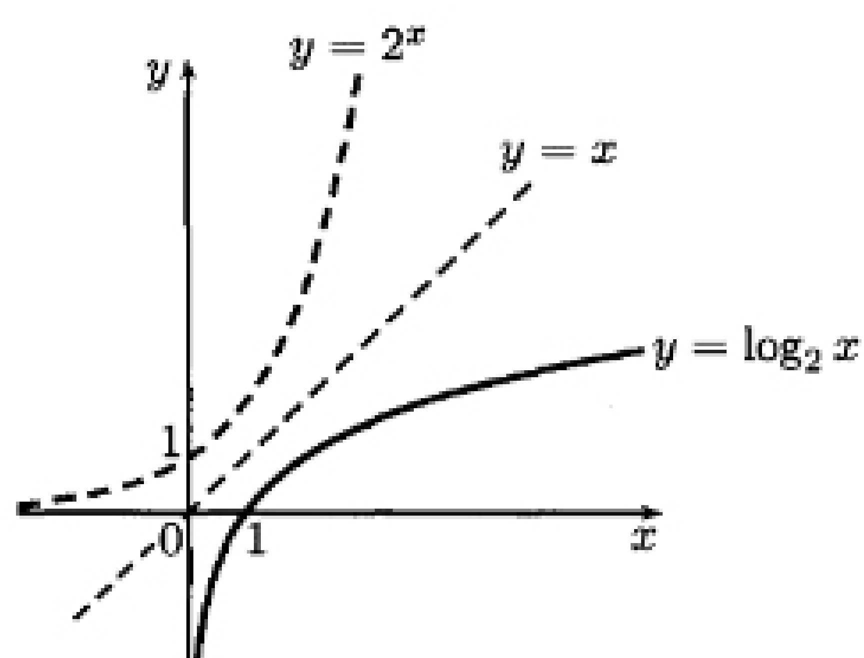
For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Remark: If a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. THUS, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$. More precisely:

Graphs of logarithmic functions:

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$.
(The picture below shows a typical case with $a > 1$.)



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Common logarithms:

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x := \log_{10} x.$$

Natural logarithms: Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition: The logarithm with base e is called the **natural logarithm** and is denoted by ln:

$$\ln x := \log_e x.$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \iff e^y = x.$$

Properties of natural logarithms:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Derivatives

Fact: By filling the table below we can convince ourselves that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	0.9516	0.9950	0.9995	0.99995	0.999995	1.000005	1.00005	1.0005	1.005	1.0517

Now, let $f(x) = e^x$. Using the definition of the derivative and the rules for exponential functions, we have that

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right) = e^x$$

Theorem:

$$\frac{d}{dx}(e^x) = e^x \qquad \text{or} \qquad (e^x)' = e^x.$$

Moreover, it follows by applying the chain rule that

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \frac{d}{dx}(g(x)) \qquad \text{or} \qquad (e^{g(x)})' = e^{g(x)}\, g'(x).$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln x$. Namely, take the derivative with respect to x of both sides of $e^{\ln x} = x$. We obtain

$$\frac{d}{dx}(e^{\ln x}) = \frac{d}{dx}(x) \qquad \text{or} \qquad e^{\ln x} \frac{d}{dx}(\ln x) = 1 \qquad \text{or} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Theorem:

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \text{or} \qquad (\ln x)' = \frac{1}{x}.$$

Moreover, it follows by applying the chain rule that

$$\frac{d}{dx}(\ln g(x)) = \frac{1}{g(x)} \frac{d}{dx}(g(x)) \qquad \text{or} \qquad (\ln g(x))' = \frac{g'(x)}{g(x)}.$$