2.3 Linear Equations

An equation $y' = f(x, y)$ is called **first-order linear** or a **linear equation** provided it can be rewritten in the special form

$$y' + p(x)y = r(x) \tag{1}$$

for some functions $p(x)$ and $r(x)$. In most applications, p and r are assumed to be continuous. The function $p(x)$ is called the **coefficient of** y. The function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

A practical test:

> An equation $y' = f(x, y)$ with f continuously differentiable is **linear** provided $f_y(x, y)$ is independent of y.

Form (1) is obtained by defining $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Two examples:

$Ly' + Ry = E$	The LR-circuit equation with $p(x) = R/L$ and $r(x) = E/L$. Symbols L, R and E are respectively inductance, resistance and electromotive force.
$y' + xy = 0$	Airy's airfoil equation with $p(x) = x$ and $r(x) = 0$.

Classifying Linear Equations

Algebraic complexity may make an equation $y' = f(x, y)$ appear to be **non-linear**, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer algebra systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Automatic simplifications in computer algebra systems make this test practical. Hand verification can use the same method.

Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x, y)$ depends on y, then $y' = f(x, y)$ is not linear.
2. If $f_{yy}(x, y) \neq 0$, then $y' = f(x, y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

The Integrating Factor Method

The initial value problem

$$(2) \qquad y' + p(x)y = r(x), \quad y(x_0) = y_0,$$

where p and r are continuous in an interval containing $x = x_0$, has an explicit solution (justified on page 78)

$$(3) \qquad y(x) = e^{-\int_{x_0}^{x} p(s)ds}\left(y_0 + \int_{x_0}^{x} r(t)e^{-\int_{x_0}^{t} p(s)ds}\,dt\right).$$

Formula (3) is called **variation of parameters**, for historical reasons. While (3) has some appeal, applications use the **integrating factor method** below, which is developed with indefinite integrals for computational efficiency. No one memorizes (3); they memorize the *method*. See Example 11, page 75, for technical details.

Integrating Factor Identity. The technique called **the method of integrating factors** uses the replacement rule (justified on page 78)

$$(4) \qquad \text{The fraction} \quad \frac{\left(e^{\int p(x)dx}Y\right)'}{e^{\int p(x)dx}} \quad \text{replaces} \quad Y' + p(x)Y.$$

The factor $e^{\int p(x)dx}$ in (4) is called an **integrating factor**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find Q	Find a simplified formula for $Q = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\frac{(Qy)'}{Q} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (4).
Method of Quadrature	Clear fractions to obtain $(Qy)' = rQ$. Apply the method of quadrature to get $Qy = \int r(x)Q(x)dx + C$. Divide by Q to isolate the explicit solution $y(x)$.

In identity (4), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. The integral $\int p(x)dx$ equals $P(x)+C$, where $P(x)$ is some anti-derivative of $p(x)$. Because $e^{\int p(x)dx} = e^{P(x)}e^{C}$, then factor e^C divides out of the fraction in (4). Applications therefore simplify the **integrating factor** $e^{\int p(x)dx}$ to $e^{P(x)}$, where $P(x)$ is *any suitable antiderivative* of $p(x)$ (effectively, we take $C = 0$).

Equation (4) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Qy)'/Q$; the method of quadrature applies to $(Qy)' = rQ$. The literature calls the exponential factor Q an **integrating factor** and equivalence (4) a **factorization** of $Y' + p(x)Y$.

Simplifying an integrating factor. Factor Q is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $Q = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$. Let $c_1 = \pm e^C$. Then $Q = c_1 Q_1$ where $Q_1 = x$. The fraction $(Qy)'/Q$ reduces to $(Q_1 y)'/Q_1$, because c_1 cancels. In an application, we choose the simpler expression Q_1. The illustration also shows that the exponential in Q can sometimes be eliminated.

Superposition

Formula (3) can be decomposed into two expressions, called y_h and y_p, so that the **general solution** is expressed as $y = y_h + y_p$. The function y_h solves the homogeneous equation $y' + p(x)y = 0$ and y_p solves the non-homogeneous equation $y' + p(x)y = r(x)$. This observation is called the **superposition principle**.

Equation (3) implies the **homogeneous solution** y_h and a **particular solution** y_p^* can be defined by

$$(5) \qquad y_h = y_0 e^{-\int_{x_0}^{x} p(s)ds}, \quad y_p^* = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t) e^{\int_{x_0}^{t} p(s)ds} dt.$$

Verification amounts to setting $r = 0$ in (3) to determine y_h. The solution y_p^* depends on the forcing term $r(x)$, but y_h does not. Initial conditions of a problem are buried in y_h. Experimentalists view the computation of y_p^* as a *single experiment* in which the state y_p^* is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$.

Structure of Solutions. Formula (3), proved on page 78, directly establishes existence for the solution to the linear initial value problem (2). The proof also determines what other particular solutions might be used in the formula for a general solution:

Theorem 2 (Solution Structure)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_h and y_p^* be defined by equation (5). Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p^*$, where $y_0 = y(x_0)$.

In short, a linear equation has the solution structure *homogeneous plus particular*. In particular, two solutions of the non-homogeneous equation differ by some solution y_h of the homogeneous equation.