Maxwell's Equations and EM Waves

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Maxwell's Equations

Let's summarize all the electromagnetic equations we have learned so far, both integral and differential forms:

1. Gauss' Law for electric field:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \qquad \text{or} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

2. Gauss' Law for magnetic field:

$$\Phi_B = \oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \qquad \text{or} \qquad \nabla \cdot \mathbf{B} = 0 \quad \text{(no magnetic monopole charges)}$$

3. Faraday's Law of Induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{A} \qquad \text{or} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

4. Ampere's Law and Maxwell's Law of Induction:

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \mathbf{E} \cdot d\mathbf{A} + \mu_0 i_{enc} \quad \text{or} \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

Derivation of Electromagnetic Wave Equation

Now let's see how we can combine the differential forms of Maxwell's equations to derive a set of differential equations (wave equations) for the electric and magnetic fields. Let's assume we solve these equations in a region without any electric charges present ($\rho=0$) or any currents ($j=0$).

Start with Maxwell's Law:

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Now take the curl of this equation:

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$$

Now it can be shown as a proof in vector calculus that:

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$

where $\nabla \cdot \mathbf{B} = 0$ by Gauss' Law for magnetic fields. And since Faraday's Law tells us:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

then we get:

$$\boxed{\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}}$$

which is a second order differential equation for <u>each</u> of the 3 components of the magnetic field. (It is a wave equation it turns out).

Now we can follow a similar derivation for the electric field starting with Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Now take the curl of this equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

Now using the same vector calculus proof:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

where $\nabla \cdot \mathbf{E} = 0$ by Gauss' Law for electric fields in vacuum. And since Maxwell's Law tells us:

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

then we get:

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

Interestingly this is the exact same differential wave equation as for magnetic fields!

Solution to Wave Equation

The general form of the wave equation is:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F}=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\mathbf{F}$$

where we have expanded the Laplacian (∇^2) operator and defined
$v=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=c=3\times10^8 \text{ m/s}$.
Solutions to this partial differential equation have the general form:

$$\mathbf{F}(\mathbf{x},t)=\mathbf{F}_m\sin(\mathbf{k}\cdot\mathbf{x}-\omega t-\phi)$$

But let's use complex notation to see how the solution works:

$$\mathbf{F}(\mathbf{x},t)=\mathbf{F}_m\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t)\right]$$

where we just take the imaginary or real part to get the physical solution.

Plugging into the wave equation yields:

$$i^2\left(k_x{}^2+k_y{}^2+k_z{}^2\right)\mathbf{F}(\mathbf{x},t)=\frac{1}{v^2}(-i\omega)^2\,\mathbf{F}(\mathbf{x},t)$$

$$\Rightarrow k^2=\frac{\omega^2}{v^2}$$

$$\boxed{v=\frac{\omega}{k}}$$

Thus, the solution to the wave equation that is a consequence of Maxwell's equations in vacuum is a sinusoidally varying function for both the electric and magnetic fields. It is a traveling wave solution, which becomes more apparent if we write the solution in this form:

$$\mathbf{E}(\mathbf{x},t)=\mathbf{E}_m\exp\left[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)\right]\quad\left(\text{or }\mathbf{E}_m\sin\left[\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)\right]\right)$$

$$\mathbf{B}(\mathbf{x},t)=\mathbf{B}_m\exp\left[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)\right]\quad\left(\text{or }\mathbf{B}_m\sin\left[\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)\right]\right)$$

Here: